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Fluctuation–dissipation relations in critical coarsening: crossover from unmagnetized to magnetized initial states

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Abstract

We study the non-equilibrium dynamics of the spherical ferromagnet quenched to its critical temperature, as a function of the magnetization of the initial state. The two limits of unmagnetized and fully magnetized initial conditions can be understood as corresponding to times that are respectively much shorter and much longer than a magnetization timescale, as in a recent field theoretical analysis of the n -vector model. We calculate exactly the crossover functions interpolating between these two limits, for the magnetization correlator and response and the resulting fluctuation–dissipation ratio (FDR). For $d > 4$ our results match those obtained recently from a Gaussian field theory. For $d < 4$, non-Gaussian fluctuations arising from the spherical constraint need to be accounted for. We extend our framework from the fully magnetized case to achieve this, providing an exact solution for the relevant integral kernel. The resulting crossover behaviour is very rich, with the asymptotic FDR X^∞ depending non-monotonically on the scaled age of the system. This is traced back to non-monotonicities of the two-time correlator, themselves the consequence of large magnetization fluctuations on the crossover timescale. We correct a trivial error in our earlier calculation for fully magnetized initial states; the corrected FDR is *consistent* with renormalization group expansions to first order in $4 - d$ for the longitudinal fluctuations of the $O(n)$ model in the limit $n \rightarrow \infty$.

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1. Introduction

The use of fluctuation–dissipation ratios (FDR) has proved very fruitful in the last decade or so for quantifying the non-equilibrium dynamics of glasses and other systems exhibiting

aging. In the context of mean-field spin glass models with infinite-range interactions, the FDR, commonly denoted X , has been used to formulate a *generalized* fluctuation–dissipation theorem (FDT) where X is interpreted in terms of an effective temperature, $T_{\text{eff}} = T/X$ for the slow, non-equilibrated modes of the system [1]. The properties of X and T_{eff} have attracted much attention, based on the hope they might allow a generalized statistical mechanical description for a broad class of non-equilibrium phenomena [2–4].

However, the generalized FDT can be shown to hold exactly only for infinite-range models. A matter of recent intense interest has been whether the appealing features of this mean-field scenario survive in more realistic systems with finite-range interactions [2]. A class of systems that has proved useful in this context is represented by ferromagnets quenched from high temperature to the critical temperature T_c or below (see, e.g., [5–9] and the recent review [10]). The non-equilibrium dynamics in these systems is due to coarsening, i.e. the growth of domains with the equilibrium magnetization (for $T < T_c$) or equilibrium correlation structure (for $T = T_c$), and slows down as domain sizes increase. In an infinite system, equilibrium is never reached, leading to aging; the age-dependence of two-time quantities has a simple physical interpretation in terms of the growth of the domain lengthscale [11]. Coarsening systems therefore provide a physically intuitive setting for the study of aging phenomena as observed, e.g. in glasses, polymers and colloids. They are, of course, not completely generic; compared to, e.g., glasses they lack features such as thermal activation over energetic or entropic barriers.

We focus in this paper on critical coarsening, i.e. coarsening at T_c , where interesting connections to dynamical universality exist. The FDR X is determined from correlation and response functions which, in aging systems, depend on two times: the age t_w of the system and a later measurement time t . In contrast to mean-field spin glasses, where X is constant within each ‘time sector’ (e.g., $t - t_w = \mathcal{O}(1)$ versus $t - t_w$ growing with t_w), in critical coarsening the FDR is a smooth function of t/t_w . This makes the interpretation of T/X as an effective temperature less obvious. To eliminate the time dependence one can consider the limit of times that are both large and well-separated. This defines an *asymptotic FDR*

$$X^\infty = \lim_{t_w \rightarrow \infty} \lim_{t \rightarrow \infty} X(t, t_w). \quad (1)$$

An important property of this quantity is that it should be *universal* [5, 10] in the sense that its value is the same for different systems falling into the same universality class of critical non-equilibrium dynamics. This makes a study of X^∞ interesting in its own right, even without an interpretation in terms of effective temperatures.

An intriguing theoretical question which has been addressed recently is whether different *initial conditions* can lead to different universality classes of critical coarsening. Due to the universality of X^∞ , these can be uncovered by studying the effect that different initial conditions have on the FDR. Of particular interest has been the effect of an initial magnetization on the ensuing coarsening. For the Ising model in high dimension or with long-range interactions [12], one finds that magnetized initial states do produce a different value of X^∞ . This suggests a different dynamical universality class from conventional coarsening from unmagnetized states, even though the magnetization decays to zero at long times. Further steps in this direction were taken in our recent calculation of exact FDRs for magnetized coarsening below the upper critical dimension in the spherical model [13]. The propagation of a trivial error meant that the results were at variance with the renormalization group (RG) result of [14] derived for the longitudinal fluctuations in the $n \rightarrow \infty$ limit of the $O(n)$ model within an expansion around $d = 4$. We give the corrected results in this paper, and these are consistent with the RG calculations (see appendix A). This suggests that the equivalence

between the dynamics of the spherical model and the large- n limit of the $O(n)$ model extends beyond the regime of Gaussian fluctuations, where it is trivial to establish.

Recently it was emphasized in the context of a field-theoretic analysis [15] that one should think of the nonzero initial magnetization as introducing a new timescale in the system. The two limits of unmagnetized and magnetized initial conditions can then be understood as corresponding to times that are respectively much shorter and much longer than this magnetization timescale, and one can in fact interpolate between these two limits using a crossover function that depends on times scaled by the magnetization timescale. This crossover function was calculated in [15] in the classical (Gaussian) regime, i.e. above the upper critical dimension, but so far there are no predictions for this function for lower dimensions where the critical behaviour is governed by non-mean-field exponents. We provide the first results of this kind in this work by calculating the relevant crossover functions exactly for the spherical model in $2 < d < 4$, for the correlator, response and FDR of the magnetization.

In section 2, we recall the known crossover behaviour of the magnetization (which is directly related to a function $g(t)$) and the general relations encoding the consequences of this for the magnetization correlation and response functions. As in [13] non-Gaussian spin fluctuations are important and will be accounted for via the kernel L . Key to our analysis for the more complicated functions $g(t)$ in our current scenario is an *exact* solution of the integral equation defining L that applies independently of the time regime. In section 3, we then evaluate the magnetization correlator and response for $d > 4$. As expected, we find here full agreement with the Gaussian field-theoretic calculations [15]. Section 4 deals with the more interesting case $d < 4$. Here the analysis is more complicated but we can still derive exact results for the asymptotic FDR X^∞ . The relevant crossover functions display unexpected non-monotonicities that, close to the lower critical dimension $d = 2$, turn into singularities at intermediate values of the scaled system age. We study carefully the relevant scaling regimes for $d \rightarrow 2$, and investigate how they arise from the behaviour of the two-time magnetization correlator. Our results are summarized in section 5.

2. Setup of calculation and exact solution for $L^{(2)}$

We start by recapitulating briefly the relevant elements of our previous analysis of critical coarsening in the spherical ferromagnet [13]. The model consists of N spins S_i on a d -dimensional cubic lattice, with sites \mathbf{r}_i and Hamiltonian $H = \frac{1}{2} \sum_{(ij)} (S_i - S_j)^2$ [16]. The spins are real valued but subject to the spherical constraint $\sum_i S_i^2 = N$. Langevin dynamics leads to a simple equation of motion for the Fourier components $S_{\mathbf{q}} = \sum_i S_i \exp(-i\mathbf{q} \cdot \mathbf{r}_i)$ of the spins, $\partial_t S_{\mathbf{q}} = -(\omega_{\mathbf{q}} + z(t))S_{\mathbf{q}} + \xi_{\mathbf{q}}$ where $\omega_{\mathbf{q}} = 2 \sum_{a=1}^d (1 - \cos q_a)$ is abbreviated to ω below and $\xi_{\mathbf{q}}$ is independent Gaussian noise on each wavevector $\mathbf{q} = (q_1, \dots, q_d)$, with $\langle \xi_{\mathbf{q}}(t) \xi_{\mathbf{q}'}^*(t') \rangle = 2NT \delta(t - t')$. The Lagrange multiplier $z(t)$ enforces the spherical constraint; as explained in [13], it is in reality not just a simple function of time but a dynamical variable with fluctuations of $\mathcal{O}(N^{-1/2})$ that cause all the non-trivial effects in the behaviour of global observables. In terms of the function $g(t) = \exp(2 \int_0^t dt' z(t'))$ the Fourier-mode response is

$$R_{\mathbf{q}}(t, t_w) = \sqrt{\frac{g(t_w)}{g(t)}} e^{-\omega(t-t_w)} = \frac{m(t)}{m(t_w)} e^{-\omega(t-t_w)}. \quad (2)$$

In the second equality we have used that the time-dependent magnetization can be written as $m(t) = (1/N) \langle S_0(t) \rangle = R_0(t, 0) (1/N) \langle S_0(0) \rangle = m_0 / \sqrt{g(t)}$ with $m_0 = (1/N) \langle S_0(0) \rangle$ the initial magnetization. The full, unsubtracted two-time correlator $C_{\mathbf{q}}(t, t_w) = (1/N) \langle S_{\mathbf{q}}(t) S_{\mathbf{q}}^*(t_w) \rangle$ can be related to its equal-time value by the response function,

$$C_{\mathbf{q}}(t, t_w) = R_{\mathbf{q}}(t, t_w)C_{\mathbf{q}}(t_w, t_w). \quad (3)$$

The relevant equal-time value is given by

$$C_{\mathbf{q}}(t, t) = \frac{C_{\mathbf{q}}(0, 0)}{g(t)} e^{-2\omega t} + 2T \int_0^t dt' \frac{g(t')}{g(t)} e^{-2\omega(t-t')}. \quad (4)$$

The function $g(t)$ is determined from the spherical constraint, which imposes $\int (dq) C_{\mathbf{q}}(t, t) = 1$. Here and below we abbreviate $(dq) \equiv d\mathbf{q}/(2\pi)^d$, where the integral runs over the first Brillouin zone of the hypercubic lattice, i.e. $\mathbf{q} \in [-\pi, \pi]^d$. The resulting integral equation for $g(t)$ is

$$g(t) = \int (dq) C_{\mathbf{q}}(0, 0) e^{-2\omega t} + 2T \int_0^t dt' g(t') f(t-t') \quad (5)$$

with $f(t) = \int (dq) e^{-2\omega t}$. Our first task will be to understand how the solution of this crosses over between the magnetized and unmagnetized cases. In terms of the Laplace transform $\hat{g}(s) = \int_0^\infty dt g(t) e^{-st}$, equation (5) reads

$$\hat{g}(s) = \frac{1}{1 - 2T \hat{f}(s)} \int (dq) \frac{C_{\mathbf{q}}(0, 0)}{s + 2\omega}. \quad (6)$$

We take as the initial condition the standard choice [15, 17] of a small magnetization m_0 but otherwise uncorrelated spin fluctuations. The initial equal-time Fourier-mode correlator can then be written as

$$C_{\mathbf{q}}(0, 0) = \delta_{\mathbf{q}, \mathbf{0}} N m_0^2 + (1 - m_0^2). \quad (7)$$

This unsubtracted correlator is $\mathcal{O}(1)$ for $\mathbf{q} \neq \mathbf{0}$ but $\mathcal{O}(N)$ for $\mathbf{q} = \mathbf{0}$. (For the fluctuation–dissipation behaviour we will need to look at the connected correlator $\tilde{C}_{\mathbf{q}}$, which is discussed below.) Equation (7) yields, bearing in mind that the integral (dq) is really a sum over the N discrete wavevectors with weight $1/N$ each,

$$\int (dq) \frac{C_{\mathbf{q}}(0, 0)}{s + 2\omega} = \frac{m_0^2}{s} + (1 - m_0^2) \hat{f}(s). \quad (8)$$

Using this in (6) one has at criticality, where $T = T_c = [\int (dq) 1/\omega]^{-1} = [2\hat{f}(0)]^{-1}$,

$$\hat{g}(s) = \hat{K}_{\text{eq}}^{-1}(s) \frac{1}{s} \left[\frac{m_0^2}{s} + (1 - m_0^2) \hat{f}(s) \right] \quad (9)$$

with

$$\hat{K}_{\text{eq}}(s) = T_c \int (dq) \frac{1}{\omega(2\omega + s)} \quad (10)$$

the Laplace transform of the equilibrium form (38) of the kernel K defined below. As before [13] we want to look at the long-time limit of $g(t)$, corresponding to small s in (9). In this regime $\hat{K}_{\text{eq}}(s)$ is given for $d > 4$ by $\hat{K}_{\text{eq}}(s) = \hat{K}_{\text{eq}}(0) + \dots$, with the higher order corrections comprising regular terms $a_1 s + a_2 s^2 + \dots$ and a leading singular term proportional to $s^{(d-4)/2}$. For $d < 4$, on the other hand, the leading behaviour for small s is $\hat{K}_{\text{eq}}(s) = b s^{(d-4)/2}$, with b some d -dependent constant. In the remaining square bracket of (9) only the first term is present for a fully magnetized initial state ($m_0 = 1$); conversely, only the second survives for the unmagnetized case ($m_0 = 0$). To see the crossover between these limits the two terms need to be of the same order. Because we are interested in small s and $\hat{f}(0)$ is nonzero, this

implies that m_0^2 and s must be of the same order. We then find to leading order in these small quantities

$$\hat{g}(s) = \begin{cases} \hat{K}_{\text{eq}}^{-1}(0) \left[\frac{m_0^2}{s^2} + \frac{\hat{f}(0)}{s} \right] & (d > 4) \\ \frac{s^{(4-d)/2}}{b} \left[\frac{m_0^2}{s^2} + \frac{\hat{f}(0)}{s} \right] & (d < 4) \end{cases} \quad (11)$$

or in the time domain

$$g(t) = \begin{cases} \hat{K}_{\text{eq}}^{-1}(0) [m_0^2 t + \hat{f}(0)] & (d > 4) \\ \frac{1}{b} \left[\frac{m_0^2}{\Gamma(d/2)} t^{(d-2)/2} + \frac{\hat{f}(0)}{\Gamma((d-2)/2)} t^{(d-4)/2} \right] & (d < 4) \end{cases} \quad (12)$$

One can combine these two expressions as

$$g(t) = \frac{1}{\mu_d} t^{-\kappa} (m_0^2 t + c) = \frac{c}{\mu_d} t^{-\kappa} \left(\frac{t}{\tau_m} + 1 \right), \quad (13)$$

where we have defined

$$\kappa = \begin{cases} \frac{4-d}{2} & (d < 4) \\ 0 & (d > 4) \end{cases} \quad \mu_d = \begin{cases} b\Gamma(d/2) & (d < 4) \\ \hat{K}_{\text{eq}}(0) & (d > 4) \end{cases} \quad (14)$$

and $c = (1 - \kappa)\hat{f}(0)$. In the second equality of (13) we have taken out the factor of c to identify the crossover timescale

$$\tau_m = \frac{c}{m_0^2}, \quad (15)$$

which as anticipated in the introduction depends on the initial magnetization of the system. In the time domain, our statement of the relevant long-time scaling $m_0^2 \sim s$ can now be phrased as follows: we will be considering the limit of large t , t_w and τ_m (corresponding to small m_0) at fixed time ratios $u_w = t_w/\tau_m$ and $u_t = t/\tau_m$. For ease of comparison with the work of [15] we will write simply $u \equiv u_w$ and mostly work with u and the time ratio $x = t/t_w = u_t/u$ instead of u and u_t . In terms of these variables one can write the function $g(t)$ as

$$g(t) = \frac{c}{\mu_d} t^{-\kappa} (ux + 1). \quad (16)$$

For the magnetization one then finds

$$m(t) = \frac{m_0}{\sqrt{g(t)}} = \sqrt{\frac{c}{\tau_m} \frac{\mu_d}{c} \frac{t^{\kappa/2}}{\sqrt{ux+1}}} = \frac{\mu_d^{1/2}}{t^{\alpha/2}} \sqrt{\frac{ux}{ux+1}} \quad (17)$$

with the exponent α defined as $\alpha = 1 - \kappa$ as in [13]. The last square root equals unity for long times if the initial magnetization is kept finite and nonzero (so that $u \gg 1$). Otherwise it gives the well-known correction to the fully magnetized result when the initial magnetization is small, i.e. when $t \sim \tau_m$ [17]. In particular, for $u_t = ux \ll 1$, the magnetization displays critical initial slip, increasing as $m(t) \sim t^{\kappa/2}$, before crossing over to the $t^{-\alpha/2}$ decay around $u_t = 1$. Our analysis for the fully magnetized case in [13] is now recognized as relating to the limit $t, t_w \gg \tau_m$, and accordingly all results in this paper should reduce to those in [13] in the limit $u \rightarrow \infty$. (Loosely speaking, one can think of this limit as corresponding to $\tau_m \rightarrow 0$, i.e. ‘ $m_0 = \infty$ ’ [15].) In the opposite limit $t, t_w \ll \tau_m$ we should get back the results for the unmagnetized case $m_0 = 0$. In terms of our scaling variables, this limit corresponds to $u \rightarrow 0$

at fixed x . Note that there is in principle a third, ‘mixed’ regime where the earlier time $t_w \ll \tau_m$ but the later time $t \gg \tau_m$, i.e. $u \ll 1$ and $ux \gg 1$. We will see, however, that essentially no new behaviour arises here and the crossover between the magnetized and unmagnetized cases, which the analysis below will allow us to elucidate explicitly, is governed principally by u .

We next explore how the crossover effects in $g(t)$ modify the expressions for the long-time behaviour of the connected Fourier-mode correlator $\tilde{C}_{\mathbf{q}}(t, t_w) = C_{\mathbf{q}}(t, t_w) - (1/N)\langle S_{\mathbf{q}}(t) \rangle \langle S_{\mathbf{q}}^*(t_w) \rangle = C_{\mathbf{q}}(t, t_w) - N\delta_{\mathbf{q}, \mathbf{0}} m(t)m(t_w)$ and the response function $R_{\mathbf{q}}(t, t_w)$. (Here, as previously, we will not write explicitly the dependence on τ_m .) From [13] we know that the equal-time connected correlator has the same expression as the unsubtracted correlator

$$\tilde{C}_{\mathbf{q}}(t_w, t_w) = \frac{1}{g(t_w)} \left[\tilde{C}_{\mathbf{q}}(0, 0) e^{-2\omega t_w} + 2T_c \int_0^{t_w} dt' e^{-2\omega(t_w-t')} g(t') \right] \quad (18)$$

except for the appropriately modified initial condition $\tilde{C}_{\mathbf{q}}(0, 0) = 1 - m_0^2$ which—in contrast to the unsubtracted $C_{\mathbf{q}}$ —is $\mathcal{O}(1)$ for all \mathbf{q} . For the zero Fourier mode one sees that in the long-time limit the first term is subleading and the integral diverges at the upper end so that one can use the asymptotics of $g(t')$, giving

$$\tilde{C}_{\mathbf{0}}(t_w, t_w) = \frac{1}{g(t_w)} \left[\tilde{C}_{\mathbf{0}}(0, 0) + 2T_c \int_0^{t_w} dt' g(t') \right] = 2T_c t_w \frac{u/(2-\kappa) + 1/(1-\kappa)}{u+1}. \quad (19)$$

Similarly in the ratio of nonzero- and zero-mode correlators, expressed in terms of the scaling variable $w = \omega t_w$,

$$\frac{\tilde{C}_{\mathbf{q}}(t_w, t_w)}{\tilde{C}_{\mathbf{0}}(t_w, t_w)} = \frac{(1 - m_0^2) e^{-2w} + 2T_c t_w \int_0^1 dz e^{-2w(1-y)} g(zt_w)}{1 + 2T_c t_w \int_0^1 dz g(zt_w)}. \quad (20)$$

one can neglect the non-integral terms for long times and gets

$$\frac{\tilde{C}_{\mathbf{q}}(t_w, t_w)}{\tilde{C}_{\mathbf{0}}(t_w, t_w)} = \frac{\int_0^1 dz e^{-2w(1-y)} z^{-\kappa} (zu + 1)}{\int_0^1 dz z^{-\kappa} (uz + 1)} = \frac{\int_0^1 dz e^{-2w(1-z)} z^{-\kappa} (uz + 1)}{u/(2-\kappa) + 1/(1-\kappa)}. \quad (21)$$

Putting the last two results together yields the general scaling

$$\tilde{C}_{\mathbf{q}}(t_w, t_w) = \frac{T_c}{\omega} \mathcal{F}_C(w, u), \quad \mathcal{F}_C(w, u) = \frac{2w}{u+1} \int_0^1 dz e^{-2w(1-z)} z^{-\kappa} (uz + 1). \quad (22)$$

One checks easily that $\mathcal{F}_C(w, u)$ reduces to the analogous scaling functions for the unmagnetized and fully magnetized cases [13] in the appropriate limits $u \rightarrow 0$ and $u \rightarrow \infty$. The magnetization response function is the zero-mode response $R_{\mathbf{0}}$. From (2), using the scaling of the magnetization found in (17), it is given by

$$R_{\mathbf{0}}(t, t_w) = \frac{m(t)}{m(t_w)} = x^{\kappa/2} \sqrt{\frac{u+1}{ux+1}}. \quad (23)$$

The results above are valid within the Gaussian approximation for the spin dynamics in the spherical model, where the small fluctuations in the Lagrange multiplier $z(t)$ are neglected. As we saw in [13], in order to study the FD behaviour of the magnetization (i.e. of the zero Fourier mode, which is a global observable) when an initial nonzero magnetization is present, we need to account for non-Gaussian corrections arising from these Lagrange multiplier fluctuations. Fortunately our earlier expressions [13] for the resulting magnetization correlator and response are valid for arbitrary initial conditions and can be used directly. The magnetization correlator including non-Gaussian effects is [13]

$$C(t, t_w) = C^{(1)}(t, t_w) + C^{(2)}(t, t_w) \quad (24)$$

with

$$C^{(1)}(t, t_w) = \tilde{C}_0(t, t_w) - \int dt' [M(t, t') \tilde{C}_0(t_w, t') + M(t_w, t') \tilde{C}_0(t, t')] m(t') \\ + \int dt' dt'_w M(t, t') M(t_w, t'_w) m(t') m(t'_w) \tilde{C}_0(t', t'_w) \quad (25)$$

$$= \int dt' dt'_w [\delta(t - t') - M(t, t') m(t')] [\delta(t_w - t'_w) - M(t_w, t'_w) m(t'_w)] \tilde{C}_0(t', t'_w) \quad (26)$$

and

$$C^{(2)}(t, t_w) = \frac{1}{2} \int dt' dt'_w M(t, t') M(t_w, t'_w) \tilde{C} \tilde{C}(t', t'_w), \quad (27)$$

where $\tilde{C} \tilde{C}(t', t'_w) = \int (dq) \tilde{C}_q^2(t', t'_w)$. The corresponding expression for the global magnetization response including non-Gaussian effects is [13]

$$R(t, t_w) = \int dt' [\delta(t - t') - M(t, t') m(t')] R_0(t', t_w). \quad (28)$$

The key function M appearing here is defined as follows. One starts from the kernel

$$K(t, t_w) = \int (dq) R_q(t, t_w) C_q(t, t_w) = \int (dq) R_q^2(t, t_w) C_q(t_w, t_w) \quad (29)$$

and its inverse L defined by

$$\int dt' K(t, t') L(t', t_w) = \delta(t - t_w). \quad (30)$$

The behaviour of $K(t, t_w)$ near $t_w = t$ can be shown to imply the following structure for L :

$$L(t, t_w) = \delta'(t - t_w) + 2T_c \delta(t - t_w) - L^{(2)}(t, t_w), \quad (31)$$

where the first term arises from the fact that $K(t, t_w)$ is causal (i.e. it vanishes for $t_w > t$) and has a unit jump at $t_w = t$. Finally, M is defined to be proportional to the integral of L ,

$$M(t, t_w) = m(t) \int^t dt' L(t', t_w). \quad (32)$$

In our previous analysis [13] we had found long-time scaling forms of $L^{(2)}$ separately for the unmagnetized and magnetized cases, with different methods needed for $d > 4$ and $d < 4$. With the function $g(t)$ no longer being a simple power law, it seems difficult if not impossible to adapt these methods to our current crossover calculation. Fortunately, however, there is a general and fully exact solution for $L^{(2)}$ which applies in any dimension and for any $g(t)$. To obtain this, we essentially integrate by parts in (30). In the derivative of K with respect to the earlier time argument we separate off the contribution from the unit step and write

$$\partial_{t_w} K(t, t_w) = -\delta(t - t_w) + K'(t, t_w), \quad (33)$$

where K' , like K , vanishes for $t_w > t$ and is finite elsewhere. Correspondingly we split off the first term from (31) and write

$$\int^t dt' L(t', t_w) = \delta(t - t_w) + N(t, t_w), \quad (34)$$

where explicitly

$$N(t, t_w) = 2T_c - \int_{t_w}^t dt' L^{(2)}(t', t_w) \quad (35)$$

and $N(t, t_w)$ also vanishes for $t_w > t$. Integrating by parts in (30) and substituting these definitions then yields

$$K'(t, t_w) + \int_{t_w}^t dt' K'(t, t') N(t', t_w) - N(t, t_w) = 0. \quad (36)$$

The point of this transformation is that non-equilibrium effects manifest themselves in K' in a very simple form. To see this, note from (4) for the unsubtracted correlator that $\partial_{t_w} C_q(t_w, t_w) = -[g'(t_w)/g(t_w) + 2\omega]C_q(t_w, t_w) + 2T_c$, while from (2) $\partial_{t_w} R_q^2(t, t_w) = [g'(t_w)/g(t_w) + 2\omega]R_q^2(t, t_w)$. Inserting into (29) gives

$$K'(t, t_w) = 2T_c \int (dq) R_q^2(t, t_w) = \frac{g(t_w)}{g(t)} 2T_c \int (dq) e^{-2\omega(t-t_w)} = -\frac{g(t_w)}{g(t)} K'_{\text{eq}}(t - t_w), \quad (37)$$

where

$$K_{\text{eq}}(t - t_w) = \int (dq) \frac{T_c}{\omega} e^{-2\omega(t-t_w)} \quad (38)$$

(with Laplace transform given by (10)) is the equilibrium form of $K(t, t_w)$. With the simple multiplicative structure of (37) one can now solve the integral equation (36) for N by inspection,

$$N(t, t_w) = N_{\text{eq}}(t - t_w) \frac{g(t_w)}{g(t)}, \quad (39)$$

where $N_{\text{eq}}(t - t_w)$ is the solution of the equilibrium version of (36), which is related to the corresponding $L_{\text{eq}}^{(2)}$ by

$$N_{\text{eq}}(t - t_w) = 2T_c - \int_0^{t-t_w} d\tau L_{\text{eq}}^{(2)}(\tau) \approx \begin{cases} \frac{2\lambda_d}{4-d} (t - t_w)^{(d-4)/2} & (d < 4) \\ \frac{1}{\mu_d} + \frac{2\lambda_d}{d-4} (t - t_w)^{(4-d)/2} & (d > 4) \end{cases}. \quad (40)$$

The last approximation gives the scalings for large time differences $t - t_w$, derived from the corresponding asymptotic behaviour of $L_{\text{eq}}^{(2)}$. The latter is $L_{\text{eq}}^{(2)}(t - t_w) = \lambda_d (t - t_w)^{(d-6)/2}$ in $d < 4$ and $L_{\text{eq}}^{(2)}(t - t_w) = \lambda_d (t - t_w)^{(2-d)/2}$ in $d > 4$, with λ_d a d -dependent coefficient [13]. This behaviour can be derived from the Laplace transform of $L_{\text{eq}}^{(2)}$, which from the equilibrium versions of (30), (31) follows as

$$\hat{L}_{\text{eq}}^{(2)}(s) = s + 2T_c - 1/\hat{K}_{\text{eq}}(s). \quad (41)$$

Note that N_{eq} decays to zero for $d < 4$ because $\hat{L}_{\text{eq}}^{(2)}(0) = \int_0^\infty d\tau L_{\text{eq}}^{(2)}(\tau) = 2T_c$ exactly, while for $d > 4$ it approaches the nonzero limit $2T_c - \hat{L}_{\text{eq}}^{(2)}(0) = 1/\mu_d$ [13].

The kernel M is directly related to N from (32) and (35),

$$M(t, t_w) = m(t)[\delta(t - t_w) + N(t, t_w)] \quad (42)$$

and in our current context we do not then need to compute $L^{(2)}$ explicitly. Briefly, though, the general solution for $L^{(2)}$ is

$$L^{(2)}(t, t_w) = -\partial_t N(t, t_w) = \frac{g(t_w)}{g(t)} L_{\text{eq}}^{(2)}(t - t_w) + \frac{g'(t)g(t_w)}{g^2(t)} N_{\text{eq}}(t - t_w) \quad (43)$$

and we outline in appendix B how this retrieves all of our previous results in the appropriate limits. The key advantage of the above solution method is that it automatically accounts for all non-equilibrium effects by reducing the problem to an equilibrium calculation at criticality, where all functions depend only on time differences and the relevant integral equation can easily be solved by Laplace transform as shown in (41).

With the general solution for $M(t, t_w)$, and hence for the magnetization correlator and response, now in hand we analyse separately the cases $d > 4$ and $d < 4$.

3. Crossover behaviour in $d > 4$

We first consider the situation $d > 4$ above the upper critical dimension. We expect to find here the same results for universal quantities as in the Gaussian field theory of [15]. The zero Fourier-mode Gaussian correlator and response are obtained from (19) and (23) by setting $\kappa = 0$,

$$R_0(t, t_w) = \sqrt{\frac{u+1}{ux+1}}, \quad \tilde{C}_0(t_w, t_w) = T_c t_w \frac{u+2}{u+1}. \quad (44)$$

With these, one has from (39), (40) and (42),

$$\begin{aligned} M(t, t')m(t') &= \frac{m_0^2}{\sqrt{g(t)g(t')}} \left\{ \delta(t-t') + \frac{g(t')}{g(t)} \left[\frac{1}{\mu_d} + \frac{2\lambda_d}{d-4} (t-t')^{(4-d)/2} \right] \right\} \\ &= \frac{\mu_d u}{t_w} \frac{1}{(ux+1)^{1/2}(uy+1)^{1/2}} \\ &\quad \times \left\{ t_w^{-1} \delta(x-y) + \frac{uy+1}{ux+1} \left[\frac{1}{\mu_d} + \frac{2\lambda_d}{d-4} t_w^{(4-d)/2} (x-y)^{(4-d)/2} \right] \right\}, \end{aligned} \quad (45)$$

where we have rescaled the times with t_w and introduced the scaling variable $y = t'/t_w$. In the long-time limit the first and the third terms in the above expression are subleading for $d > 4$, so

$$M(t, t')m(t') = \frac{u(uy+1)^{1/2}}{t_w(ux+1)^{3/2}}. \quad (46)$$

By inserting this expression into (28) one finds the magnetization response

$$\begin{aligned} R(t, t_w) &= R_0(t, t_w) - \int dt' M(t, t')m(t')R_0(t', t_w) \\ &= \left(\frac{u+1}{ux+1} \right)^{1/2} - \frac{u(u+1)^{1/2}}{(ux+1)^{3/2}}(x-1) = \left(\frac{u+1}{ux+1} \right)^{3/2}. \end{aligned} \quad (47)$$

The magnetization correlator is found from (25) and reads after rescaling all times

$$\begin{aligned} C^{(1)}(t, t_w) &= T_c t_w \left\{ \frac{u+2}{u+1} \left[\sqrt{\frac{u+1}{ux+1}} - \int_1^x dy \frac{u(u+1)^{1/2}}{(ux+1)^{3/2}} \right] - \int_0^1 dy \frac{uy(uy+2)}{(ux+1)^{3/2}(u+1)^{1/2}} \right. \\ &\quad - \int_0^1 dy \frac{uy(uy+2)}{(u+1)^{3/2}(ux+1)^{1/2}} + \int_1^x dy \int_0^1 dy_w \frac{u^2 y_w (u y_w + 2)}{(ux+1)^{3/2}(u+1)^{3/2}} \\ &\quad + \int_0^1 dy \int_0^y dy_w \frac{u^2 y_w (u y_w + 2)}{(ux+1)^{3/2}(u+1)^{3/2}} \\ &\quad \left. + \int_0^1 dy \int_y^1 dy_w \frac{u^2 y (u y + 2)}{(ux+1)^{3/2}(u+1)^{3/2}} \right\} \\ &= T_c t_w \frac{2 + 3u + 2u^2 + \frac{1}{2}u^3}{(u+1)^{3/2}(ux+1)^{3/2}}. \end{aligned} \quad (48)$$

The term $C^{(2)}$ scales as $\sim t_w^{(4-d)/2}$ for $4 < d < 6$, where the integral (27) that defines it can be shown to be dominated by aging timescales, and as $\sim t_w^{-1}$ for $d > 6$, where $\tilde{C}\tilde{C}(t', t'_w)$ in (27) behaves as a short-range kernel, so it is always subleading. Thus (48) represents the full long-time magnetization correlator for $d > 4$.

The t -dependence in the correlator $C \equiv C^{(1)}$ is the same as in the response R and only occurs via the overall factor $(ux+1)^{-3/2} = (u_t+1)^{-3/2}$. It therefore cancels in the resulting

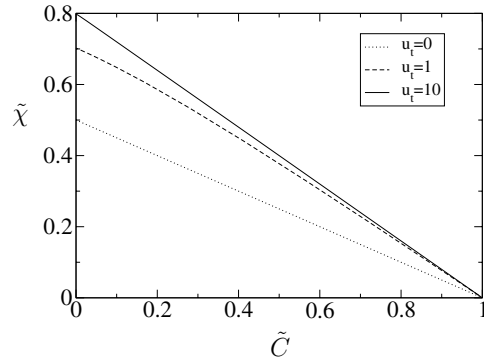


Figure 1. Normalized magnetization FD plot for dimensionality d above 4, showing the normalized susceptibility $\tilde{\chi}$ versus the normalized correlator \tilde{C} , for different fixed values of $u_t = ux$ as indicated in the figure. For $u_t = 0$ the plot is a straight line with (negative) slope $1/2$, as expected from the unmagnetized limit. As u_t is increased the initial slope of the plot converges quickly to $4/5$, corresponding to the fully magnetized limit, and the crossover to the unmagnetized regime occurs at larger time differences and eventually becomes invisible on the scale of the plot.

FDR which follows after a few lines (using $\partial_{t_w}[t_w F(x, u)] = (1 + u\partial_u - x\partial_x)F(x, u)$ to calculate $\partial_{t_w} C$) as

$$X(t, t_w) = \frac{T_c R(t, t_w)}{\partial_{t_w} C(t, t_w)} = \frac{4}{5} \frac{(u+1)^4}{(u+1)^4 + \frac{3}{5}} \equiv X^\infty(u). \quad (49)$$

Thus, for $d > 4$ the FDR is t -independent and hence identical to the asymptotic FDR $X^\infty(u) = \lim_{t \gg t_w = u\tau_m \gg 1} X(t, t_w)$. It interpolates between $1/2$ (for $u \ll 1$) and $4/5$ (for $u \gg 1$), reproducing in these limits our previous results for the FDRs for unmagnetized and fully magnetized initial conditions [13]. As expected from the universality of X^∞ , our result for the entire crossover function also exactly agrees with that calculated from a Gaussian field theory [15]. The FD plot is obtained by graphing the normalized susceptibility $\tilde{\chi}(t, t_w) = T_c \chi(t, t_w)/C(t, t)$ versus the normalized correlator $\tilde{C}(t, t_w) = C(t, t_w)/C(t, t)$ at fixed $u_t = ux$ and using x (or u) as the curve parameter. The factor of T_c is included in the definition of $\tilde{\chi}$ to make the equilibrium FD plot a line of (negative) slope 1. The susceptibility is obtained from R by integration as usual,

$$\chi(t, t_w) = \int_{t_w}^t dt' R(t, t') = \frac{t_w}{u} \int_u^{u_t} du' R(u_t, u') = \frac{2}{5} \frac{t_w}{u} \frac{(ux+1)^{5/2} - (u+1)^{5/2}}{(ux+1)^{3/2}} \quad (50)$$

(with some obvious abuse of notation in the representation as an integral over u'). The results, displayed in figure 1, show that for $u_t = 0$, the curve is a straight line with (negative) slope $1/2$, as expected from the unmagnetized limit. As u_t is increased, the initial slope of the plot converges quickly to $4/5$, which is expected in the fully magnetized regime, and the unmagnetized regime gets progressively squeezed into the top left corner of the plot where it eventually becomes invisible. Intuitively, this is because for $u_t = t/\tau_m \gg 1$ we need to move to relatively much earlier times $u = t_w/\tau_m \sim 1$ in order for the dynamics to be sensitive to the fact that the initial magnetization was small.

4. Crossover behaviour in $d < 4$

In $d < 4$ the analysis is somewhat more awkward and leads to highly non-trivial magnetization FD behaviour as we will see. As before we will find that all relevant quantities vary on aging

timescales $\sim t_w$ and so we will exploit the relevant asymptotics for large time differences throughout.

One starts by working out the combination $M(t, t_w)m(t_w)$ appearing in the definition of C and R ,

$$M(t, t_w)m(t_w) = N(t, t_w)m(t_w)m(t) = \frac{m_0^2}{\sqrt{g(t)g(t_w)}} \frac{g(t_w)}{g(t)} \frac{2\lambda_d}{4-d} (t-t_w)^{(d-4)/2} \quad (51)$$

$$= \frac{2\lambda_d \mu_d m_0^2 t_w^{-3\kappa/2}}{(4-d)c} \frac{(u+1)^{1/2}}{t^{-3\kappa/2} (ux+1)^{3/2}} (x-1)^{(d-4)/2} = \frac{1}{t} \mathcal{F}_M(x, u), \quad (52)$$

where we have defined

$$\mathcal{F}_M(x, u) = \frac{d-2}{2} ux^{(8-d)/4} \frac{(u+1)^{1/2}}{(ux+1)^{3/2}} \left(\frac{x-1}{x}\right)^{(d-4)/2} \quad (53)$$

and used $2\lambda_d \mu_d = (4-d)(d-2)/2$ [13]. Equation (53) represents the generalization to finite u of the scaling function $\mathcal{F}_M(x)$ determined in [13] and reduces to the latter in the limit $u \rightarrow \infty$ as it should. Note that in (51) we have directly neglected the contribution $\delta(t-t_w)m(t_w)m(t) \sim \delta(x-1)t_w^{-1}t_w^{-\alpha/2}t^{-\alpha/2} \sim t_w^{\kappa-2} = t_w^{-d/2}$ because it is subleading for long times compared to the main $1/t \sim 1/t_w$ term in (52).

We note briefly the explicit expression

$$\mathcal{F}_M\left(\frac{x}{y}, uy\right) = \frac{d-2}{2} ux^{4-3d/4} \frac{(uy+1)^{1/2}}{(ux+1)^{3/2}} y^{(d-4)/4} (x-y)^{(d-4)/2} \quad (54)$$

that recurs in a number of calculations below; $\mathcal{F}_M(1/y_w, uy_w)$ is obtained from this by replacing $x \rightarrow 1, y \rightarrow y_w$. It will also be useful for later to have the asymptotics of $\mathcal{F}_M(x, u)$ for large x , which will give the asymptotic behaviour of the correlator and thus of X^∞ ,

$$\mathcal{F}_M(x, u) = \frac{d-2}{2} x^{(2-d)/4} \left(\frac{u+1}{u}\right)^{1/2}. \quad (55)$$

For the response function the Gaussian contribution is from (23), after setting $\kappa = (4-d)/2$,

$$R_0(t, t_w) = x^{(4-d)/4} \sqrt{\frac{u+1}{ux+1}}. \quad (56)$$

The overall magnetization response is then found simply by inserting (53) into (28) and rescaling the times as before

$$R(t, t_w) = x^{(4-d)/4} \sqrt{\frac{u+1}{ux+1}} - \frac{d-2}{2} x^{3(4-d)/4} \frac{u(u+1)^{1/2}}{(ux+1)^{3/2}} \int_1^x dy (x-y)^{(d-4)/2} \quad (57)$$

$$= x^{(4-d)/4} \sqrt{\frac{u+1}{ux+1}} \left[1 - \frac{ux}{ux+1} \left(\frac{x-1}{x}\right)^{(d-2)/2} \right]. \quad (58)$$

For $u \gg 1$, one retrieves the fully magnetized limit calculated previously [13]. On the other hand, the unmagnetized limit, $u \ll 1$, coincides with the Gaussian response. This is consistent with the fact that non-Gaussian effects in the FD behaviour of the (global) magnetization only need to be accounted for in the case of an initial nonzero magnetization [13]. The large- x behaviour of $R(t, t_w)$ for general u , which will provide the asymptotic FDR, is easily extracted from (58) as

$$R(t, t_w) = x^{-(d+2)/4} \left(\frac{d-2}{2} + \frac{1}{u}\right) \left(\frac{u+1}{u}\right)^{1/2}. \quad (59)$$

We next turn to the correlator. Rescaling the times with t_w in equation (26) one has for the first part

$$C^{(1)}(t, t_w) = \int_0^x dy \int_0^1 dy_w \left[\delta(x-y) - \frac{1}{x} \mathcal{F}_M \left(\frac{x}{y}, uy \right) \right] \times \left[\delta(1-y_w) - \mathcal{F}_M \left(\frac{1}{y_w}, uy_w \right) \right] \tilde{C}_0(t_w y, t_w y_w). \quad (60)$$

From (3), (19) and (56) the Gaussian factor can for long times be written as

$$\tilde{C}_0(t_w y, t_w y_w) = 2T_c t_w [f(y, y_w)\theta(y - y_w) + f(y_w, y)\theta(y_w - y)] \quad (61)$$

$$= 2T_c t_w [f(y, y_w) + \theta(y_w - y)(f(y_w, y) - f(y, y_w))], \quad (62)$$

where we have explicitly accounted for the ordering of the time arguments; the dependence on y and y_w is through the function

$$f(y, y_w) = \left(\frac{y}{y_w} \right)^{\kappa/2} y_w \frac{uy_w/(2-\kappa) + 1/(1-\kappa)}{uy_w + 1} \sqrt{\frac{uy_w + 1}{uy + 1}} \quad (63)$$

$$= y^{(4-d)/4} y_w^{d/4} \frac{(2uy_w)/d + 2/(d-2)}{(uy_w + 1)^{1/2}(uy + 1)^{1/2}}. \quad (64)$$

Inserting (62) into (60), one can rewrite $C^{(1)}$ as

$$\frac{C^{(1)}(t, t_w)}{2T_c t_w} = \int_0^x dy \int_0^1 dy_w \left[\delta(x-y) - \frac{1}{x} \mathcal{F}_M \left(\frac{x}{y}, uy \right) \right] \times \left[\delta(1-y_w) - \mathcal{F}_M \left(\frac{1}{y_w}, uy_w \right) \right] f(y, y_w) - \int_0^1 dy_w \int_0^{y_w} dy \frac{1}{x} \mathcal{F}_M \left(\frac{x}{y}, uy \right) \left[\delta(1-y_w) - \mathcal{F}_M \left(\frac{1}{y_w}, uy_w \right) \right] \times [f(y_w, y) - f(y, y_w)]. \quad (65)$$

The decomposition (62) is the analogue of the cancellation trick used for the fully magnetized case [13]. There the analogue of the first integral in (65) vanished identically. This is not the case here, but the procedure remains useful because it makes it easier to extract the large- x limit: the first integral (denoted F below) factorizes, and in the second one (denoted S) both integration variables y, y_w are ≤ 1 and so $\ll x$ for large x . Using the factorization, the first double integral can be worked out explicitly for generic x ,

$$F = \int_0^x dy \left[\delta(x-y) - \frac{1}{x} \mathcal{F}_M \left(\frac{x}{y}, uy \right) \right] \frac{y^{(4-d)/4}}{(uy + 1)^{1/2}} \times \int_0^1 dy_w \left[\delta(1-y_w) - \mathcal{F}_M \left(\frac{1}{y_w}, uy_w \right) \right] y_w^{d/4} \frac{(2uy_w)/d + 2/(d-2)}{(uy_w + 1)^{1/2}} \quad (66)$$

$$= \frac{x^{(4-d)/4}}{(ux + 1)^{1/2}} \left[1 - \frac{d-2}{2} x^{(4-d)/2} \frac{u}{ux + 1} \int_0^x dy (x-y)^{(d-4)/2} \right] \times \int_0^1 dy_w \left[\delta(1-y_w) - \mathcal{F}_M \left(\frac{1}{y_w}, uy_w \right) \right] y_w^{d/4} \frac{(2uy_w)/d + 2/(d-2)}{(uy_w + 1)^{1/2}} \quad (67)$$

$$= \frac{x^{(4-d)/4}}{(ux + 1)^{3/2}} \left[\frac{2u/d + 2/(d-2)}{(u+1)^{1/2}} - \frac{d-2}{2} \frac{u}{(u+1)^{3/2}} \int_0^1 dy_w y_w^{(d-2)/2} \left(\frac{2uy_w}{d} + \frac{2}{d-2} \right) (1-y_w)^{(d-4)/2} \right]. \quad (68)$$

The remaining integral produces Beta functions so that

$$F = \frac{x^{(4-d)/4}}{(ux+1)^{3/2}} \left\{ \frac{2u/d + 2/(d-2)}{(u+1)^{1/2}} - \frac{u}{(u+1)^{3/2}} \frac{\Gamma^2(d/2)}{\Gamma(d)} \left[u + \frac{2(d-1)}{d-2} \right] \right\}. \quad (69)$$

The large- x behaviour is obtained by replacing the prefactor with $x^{-(2+d)/4}/u^{3/2}$. The second double integral in (65) can be written explicitly as

$$\begin{aligned} S = & -\frac{1}{x} \int_0^1 dy \mathcal{F}_M \left(\frac{x}{y}, uy \right) \frac{1}{(uy+1)^{1/2}(u+1)^{1/2}} \\ & \times \left[y^{d/4} \left(\frac{2uy}{d} + \frac{2}{d-2} \right) - y^{(4-d)/4} \left(\frac{2u}{d} + \frac{2}{d-2} \right) \right] \\ & + \frac{1}{x} \int_0^1 dy_w \mathcal{F}_M \left(\frac{1}{y_w}, uy_w \right) \frac{y_w^{(4-d)/4}}{(uy_w+1)^{1/2}} \\ & \times \int_0^{y_w} dy \mathcal{F}_M \left(\frac{x}{y}, uy \right) \frac{y^{d/4}}{(uy+1)^{1/2}} \left(\frac{2uy}{d} + \frac{2}{d-2} \right) \\ & - \frac{1}{x} \int_0^1 dy_w \mathcal{F}_M \left(\frac{1}{y_w}, uy_w \right) \frac{y_w^{d/4}}{(uy_w+1)^{1/2}} \left(\frac{2uy_w}{d} + \frac{2}{d-2} \right) \\ & \times \int_0^{y_w} dy \mathcal{F}_M \left(\frac{x}{y}, uy \right) \frac{y^{(4-d)/4}}{(uy+1)^{1/2}} \end{aligned} \quad (70)$$

$$\begin{aligned} = & -\frac{d-2}{2} \frac{x^{3-3d/4}u}{(u+1)^{1/2}(ux+1)^{3/2}} \int_0^1 dy (x-y)^{(d-4)/2} y^{(d-4)/4} \\ & \times \left[y^{d/4} \left(\frac{2uy}{d} + \frac{2}{d-2} \right) - y^{(4-d)/4} \left(\frac{2u}{d} + \frac{2}{d-2} \right) \right] \\ & + \left(\frac{d-2}{2} \right)^2 \frac{x^{3-3d/4}u^2}{(u+1)^{3/2}(ux+1)^{3/2}} \\ & \times \left[\int_0^1 dy_w (1-y_w)^{(d-4)/2} \int_0^{y_w} dy (x-y)^{(d-4)/2} y^{(d-2)/2} \left(\frac{2uy}{d} + \frac{2}{d-2} \right) \right. \\ & \left. - \int_0^1 dy_w (1-y_w)^{(d-4)/2} y_w^{(d-2)/2} \left(\frac{2uy_w}{d} + \frac{2}{d-2} \right) \int_0^{y_w} dy (x-y)^{(d-4)/2} \right]. \end{aligned} \quad (71)$$

If we can now take the large- x limit, where $(x-y)^{(d-4)/2} \approx x^{(d-4)/2}$, the integrals can be carried out and the prefactor simplifies, giving after a little algebra,

$$S = \frac{x^{-(2+d)/4}}{u^{3/2}(u+1)^{3/2}} \left[\frac{d-2}{d} u(u+1) \left(\frac{d}{d+2} u + 1 \right) - \frac{d-2}{2} \frac{\Gamma^2(d/2)}{\Gamma(d)} u^2 \left(\frac{u}{2} + 1 \right) \right]. \quad (72)$$

Gathering the contributions from (69) and (72), one gets for the large- x limit of the first contribution to the correlator

$$\frac{C^{(1)}(t, t_w)}{2T_c t_w} = \frac{x^{-(d+2)/4}}{u^{3/2}(u+1)^{3/2}} (Au^3 + Bu^2 + Cu + D) \quad (73)$$

with

$$A = \frac{d-2}{d+2} - \frac{d-2}{4} \frac{\Gamma^2(d/2)}{\Gamma(d)} \quad (74)$$

$$B = \frac{2d}{d+2} - \frac{d}{2} \frac{\Gamma^2(d/2)}{\Gamma(d)} \quad (75)$$

$$C = \frac{d}{d-2} - \frac{2(d-1)}{d-2} \frac{\Gamma^2(d/2)}{\Gamma(d)} \tag{76}$$

$$D = \frac{2}{d-2}. \tag{77}$$

The second contribution to the magnetization correlation comes from (27), and to make progress here we need the long-time behaviour of the two-time function $\tilde{C}\tilde{C}$ for the current case $d < 4$. Proceeding as for the fully magnetized scenario [13], we find first the scaling of the equal-time value $\tilde{C}\tilde{C}(t_w, t_w) = \int (dq) \tilde{C}_q^2(t_w, t_w) = T_c^2 \int (dq) \omega^{-2} \mathcal{F}_C(\omega t_w, u)$. This is dominated by small ω , where $(dq) = \sigma_d d\omega \omega^{(d-2)/2}$ with σ_d the surface area of a unit sphere in d dimensions. Rescaling to $w = \omega t_w$ gives

$$\tilde{C}\tilde{C}(t_w, t_w) = \gamma_d(u) t_w^{(4-d)/2}, \quad \gamma_d(u) = T_c^2 \sigma_d \int dw w^{(d-6)/2} \mathcal{F}_C^2(w, u). \tag{78}$$

Normalizing $\tilde{C}\tilde{C}(t', t'_w)$ with the equal time value $\tilde{C}\tilde{C}(t', t')$, one obtains for $t' > t'_w$ in terms of the scaling variables $y = t'/t_w$ and $y_w = t'_w/t_w$ (by rescaling in the numerator to $w = \omega t'_w$ and in the denominator to $w = \omega t'$),

$$\frac{\tilde{C}\tilde{C}(t', t'_w)}{\tilde{C}\tilde{C}(t', t')} = \frac{g(t'_w) t_w^{(4-d)/2} \int dw w^{(d-6)/2} e^{-2w(y/y_w-1)} \mathcal{F}_C^2(w, uy_w)}{g(t') t'^{(4-d)/2} \int dw w^{(d-6)/2} \mathcal{F}_C^2(w, uy)} \tag{79}$$

$$= \frac{uy_w + 1}{uy + 1} \frac{\int dw w^{(d-6)/2} e^{-2w(y/y_w-1)} \mathcal{F}_C^2(w, uy_w)}{\int dw w^{(d-6)/2} \mathcal{F}_C^2(w, uy)}. \tag{80}$$

For $t' < t'_w$, on the other hand, one has

$$\frac{\tilde{C}\tilde{C}(t', t'_w)}{\tilde{C}\tilde{C}(t', t')} = \frac{\tilde{C}\tilde{C}(t'_w, t')}{\tilde{C}\tilde{C}(t'_w, t'_w)} \frac{y_w^{(4-d)/2} \gamma_d(uy_w)}{y^{(4-d)/2} \gamma_d(uy)}. \tag{81}$$

So overall

$$\frac{\tilde{C}\tilde{C}(t', t'_w)}{\tilde{C}\tilde{C}(t', t')} = \mathcal{G}\left(\frac{y}{y_w}, uy_w\right), \tag{82}$$

where

$$\mathcal{G}\left(\frac{y}{y_w}, uy_w\right) = \begin{cases} \frac{uy_w + 1}{uy + 1} \frac{\int dw w^{(d-6)/2} \mathcal{F}_C^2(w, uy_w) e^{-2(y/y_w-1)w}}{\int dw w^{(d-6)/2} \mathcal{F}_C^2(w, uy)} & \text{for } y/y_w \geq 1 \\ \frac{\gamma_d(uy_w)}{\gamma_d(uy)} \left(\frac{y}{y_w}\right)^{(d-4)/2} \mathcal{G}(y_w/y, uy) & \text{for } y_w/y \leq 1 \end{cases}. \tag{83}$$

In the limit $u \gg 1$, this function should match with $\mathcal{G}(x)$ defined in [13] for fully magnetized initial conditions, for $x = y/y_w$. Unfortunately there was a typographical error in the definition of $\mathcal{G}(x)$ as given in [13], which propagated through the remainder of the calculation. In appendix A we state the correct versions of all the relevant equations. These include, in particular, the first-order expansions around $d = 4$ and $d = 2$ of X^∞ in the fully magnetized case.

Having clarified the scaling behaviour of $\tilde{C}\tilde{C}(t', t'_w)$ in $d < 4$, we can now work out $C^{(2)}$ from (27) by multiplying and dividing by $m(t')m(t'_w)$ and using (52)

$$C^{(2)}(t, t_w) = \frac{1}{2} \int_0^t dt' \int_0^{t_w} dt'_w \frac{1}{m(t')m(t'_w)t_w} \mathcal{F}_M\left(\frac{x}{y}, uy\right) \mathcal{F}_M\left(\frac{1}{y_w}, uy_w\right) \times \tilde{C}\tilde{C}(t', t') \mathcal{G}\left(\frac{y}{y_w}, uy_w\right) \tag{84}$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^x dy \int_0^1 dy_w \frac{t_w^\alpha}{x \mu_d} (y y_w)^{\alpha/2} \frac{(u y + 1)^{1/2} (u y_w + 1)^{1/2}}{u (y y_w)^{1/2}} \\
 &\quad \times \mathcal{F}_M \left(\frac{x}{y}, u y \right) \mathcal{F}_M \left(\frac{1}{y_w}, u y_w \right) t_w^{(4-d)/2} y^{(4-d)/2} \gamma_d(u y) \mathcal{G} \left(\frac{y}{y_w}, u y_w \right) \quad (85)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \frac{t_w}{x u \mu_d} \int_0^x dy \int_0^1 dy_w y^{(4-d)/4} y_w^{(d-4)/4} (u y + 1)^{1/2} (u y_w + 1)^{1/2} \\
 &\quad \times \mathcal{F}_M \left(\frac{x}{y}, u y \right) \mathcal{F}_M \left(\frac{1}{y_w}, u y_w \right) \gamma_d(u y) \mathcal{G} \left(\frac{y}{y_w}, u y_w \right) \quad (86)
 \end{aligned}$$

$$= \frac{t_w}{x} \int_0^1 dy_w y_w (u y_w + 1)^{1/2} \mathcal{F}_M \left(\frac{1}{y_w}, u y_w \right) \times W, \quad (87)$$

where we have defined ($v = y/y_w$)

$$2u \mu_d W = \int_0^{x/y_w} dv v^{(4-d)/4} (u v y_w + 1)^{1/2} \mathcal{F}_M \left(\frac{x}{v y_w}, u v y_w \right) \gamma_d(u v y_w) \mathcal{G}(v, u y_w). \quad (88)$$

To evaluate W we split the integral into $v = 0 \dots 1$ and $v = 1 \dots \infty$. In the former regime we rewrite $\mathcal{G}(v, \dots)$ in terms of $\mathcal{G}(1/v, \dots)$ using (83) and then transform $v \rightarrow 1/v$ to get

$$\begin{aligned}
 2u \mu_d W &= \int_0^1 dv v^{(d-4)/4} (u v y_w + 1)^{1/2} \mathcal{F}_M \left(\frac{x}{v y_w}, u v y_w \right) \gamma_d(u y_w) \mathcal{G} \left(\frac{1}{v}, u v y_w \right) \\
 &\quad + \int_1^{x/y_w} dv v^{(4-d)/4} (u v y_w + 1)^{1/2} \mathcal{F}_M \left(\frac{x}{v y_w}, u v y_w \right) \gamma_d(u v y_w) \mathcal{G}(v, u y_w) \quad (89)
 \end{aligned}$$

$$\begin{aligned}
 &= \int_1^\infty dv v^{-(d+4)/4} \left(\frac{u y_w}{v} + 1 \right)^{1/2} \mathcal{F}_M \left(\frac{x v}{y_w}, \frac{u y_w}{v} \right) \gamma_d(u y_w) \mathcal{G} \left(v, \frac{u y_w}{v} \right) \\
 &\quad + \int_1^{x/y_w} dv v^{(4-d)/4} (u v y_w + 1)^{1/2} \mathcal{F}_M \left(\frac{x}{v y_w}, u v y_w \right) \gamma_d(u v y_w) \mathcal{G}(v, u y_w). \quad (90)
 \end{aligned}$$

By performing the w -integrals in (83) explicitly, having first inserted definition (22) of \mathcal{F}_C , one finds for the terms involving \mathcal{G} ,

$$\frac{\gamma_d(u y_w) \mathcal{G}(v, \frac{u y_w}{v})}{T_c^2 \sigma_d} = \frac{u y_w / v + 1}{u y_w + 1} \int dw w^{(d-6)/2} \mathcal{F}_C^2(w, u y_w / v) e^{-2w(v-1)} \quad (91)$$

$$\begin{aligned}
 &= 2^{(4-d)/2} \frac{\Gamma(d/2)}{(u y_w / v + 1)(u y_w + 1)} \int_0^1 dz dz' (z z')^{(d-4)/2} \\
 &\quad \times (v - z - z' + 1)^{-d/2} \left(\frac{u y_w z}{v} + 1 \right) \left(\frac{u y_w z'}{v} + 1 \right) \quad (92)
 \end{aligned}$$

and the replacement $u \rightarrow uv$ gives a similar expression for $\gamma_d(u v y_w) \mathcal{G}(v, u y_w) / (T_c^2 \sigma_d)$. We now insert these back into (90) to obtain

$$\begin{aligned}
 \frac{2 \mu_d u W}{T_c^2 \sigma_d} &= \frac{2^{(4-d)/2} \Gamma(d/2)}{u y_w + 1} \left[\int_1^\infty dv v^{-(d+4)/4} \frac{1}{(u y_w / v + 1)^{1/2}} \mathcal{F}_M \left(\frac{x v}{y_w}, \frac{u y_w}{v} \right) \right. \\
 &\quad \times \int_0^1 dz dz' (z z')^{(d-4)/2} (v - z - z' + 1)^{-d/2} \left(\frac{u y_w z}{v} + 1 \right) \left(\frac{u y_w z'}{v} + 1 \right) \\
 &\quad + \int_1^{x/y_w} dv v^{(4-d)/4} \frac{1}{(u y_w v + 1)^{1/2}} \mathcal{F}_M \left(\frac{x}{v y_w}, u v y_w \right) \\
 &\quad \left. \times \int_0^1 dz dz' (z z')^{(d-4)/2} (v - z - z' + 1)^{-d/2} (u y_w z + 1) (u y_w z' + 1) \right]. \quad (93)
 \end{aligned}$$

So far our calculation of $C^{(2)}$ applies for generic x ; to make more progress we consider again the large- x behaviour. In the first v -integral one can use directly the asymptotic form (55) of \mathcal{F}_M ; for the second integral one can show as in the fully magnetized case [13] that the same replacement can be made and the upper integration limit sent to infinity thereafter. This gives

$$\begin{aligned} \frac{2\mu_d W}{T_c^2 \sigma_d} &= \frac{d-2}{2} 2^{(4-d)/2} \Gamma(d/2) x^{(2-d)/4} \frac{y_w^{(d-4)/4}}{u^{3/2}(uy_w+1)} \left[\int_1^\infty dv v^{-d/2} \int_0^1 dz dz' (zz')^{(d-4)/2} \right. \\ &\quad \times (v-z-z'+1)^{-d/2} \left(\frac{uy_w z}{v} + 1 \right) \left(\frac{uy_w z'}{v} + 1 \right) \\ &\quad \left. + \int_1^\infty dv \int_0^1 dz dz' (zz')^{(d-4)/2} (v-z-z'+1)^{-d/2} (uy_w z+1) (uy_w z'+1) \right]. \end{aligned} \quad (94)$$

Then from (87) one has

$$\begin{aligned} C^{(2)}(t, t_w) &= \left(\frac{d-2}{2} \right)^2 \frac{2^{(4-d)/2} \Gamma(d/2) T_c^2 \sigma_d t_w}{u^{1/2}(u+1)^{3/2} 2\mu_d} x^{-(d+2)/4} \int_0^1 dy_w y_w^{(d-2)/2} (1-y_w)^{(d-4)/2} \\ &\quad \times \left[\int_1^\infty dv \dots + \int_1^\infty dv \dots \right], \end{aligned} \quad (95)$$

where the v -integrals are as in (94). Carrying out the y_w -integral, this can be written as

$$C^{(2)}(t, t_w) = \left(\frac{d-2}{2} \right)^2 \frac{\Gamma((d+4)/2)}{\Gamma((4-d)/2)\Gamma(d+1)} \frac{u^{3/2} t_w x^{-(d+2)/4}}{(u+1)^{3/2}} \left[V_d + \frac{1}{u} V'_d + \frac{1}{u^2} V''_d \right] \quad (96)$$

if we define V_d as in the fully magnetized case, see appendix A,

$$V_d = \int_1^\infty dv (v^{-(d+4)/2} + 1) \int_0^1 dz dz' (zz')^{(d-2)/2} (v-z-z'+1)^{-d/2} \quad (97)$$

and introduce also the analogous quantities

$$V'_d = \frac{4d}{d+2} \int_1^\infty dv (v^{-(d+2)/2} + 1) \int_0^1 dz dz' z^{(d-4)/2} z'^{(d-2)/2} (v-z-z'+1)^{-d/2} \quad (98)$$

$$V''_d = \frac{4(d-1)}{d+2} \int_1^\infty dv (v^{-d/2} + 1) \int_0^1 dz dz' (zz')^{(d-4)/2} (v-z-z'+1)^{-d/2}. \quad (99)$$

We have also used in (96) the explicit expression [13]

$$\frac{T_c^2 \sigma_d}{2\mu_d} = \frac{-2T_c}{(d-2)(4-d)2^{(2-d)/2} \Gamma((4-d)/2) \Gamma((d-4)/2) \Gamma((d-2)/2)}. \quad (100)$$

With the results (59), (73) and (96) for the magnetization response and correlation in the limit of long, well-separated ($x \gg 1$) times, we can finally compute the asymptotic FDR as

$$\begin{aligned} X^\infty &= \left(\frac{d-2}{2} u + 1 \right) (u+1)^3 \left\{ 2 \left[\left(\frac{d(u+1)}{4} - \frac{3u}{2} \right) P_3(u) + u(u+1) P'_3(u) \right] \right. \\ &\quad + \left(\frac{d-2}{2} \right)^2 \frac{\Gamma((d+4)/2)}{\Gamma((4-d)/2)\Gamma(d+1)} \\ &\quad \left. \times \left[u P_2(u) \left(\frac{(d+12)(u+1)}{4} - \frac{3u}{2} \right) - u(u+1) P_1(u) \right] \right\}^{-1}, \end{aligned} \quad (101)$$

where we have defined the following third-, second- and first-order polynomials in u :

$$\begin{aligned} P_3(u) &= Au^3 + Bu^2 + Cu + D \\ P_2(u) &= V_d u^2 + V'_d u + V''_d \\ P_1(u) &= V'_d u + 2V''_d. \end{aligned} \quad (102)$$

The general structure of the asymptotic FDR is thus as for $d > 4$, i.e. a ratio of fourth-order polynomials in u . One can easily check that as $d \rightarrow 4$ the coefficients continuously approach those for $d > 4$, as they should. Also, the $u \ll 1$ -limit of (101) retrieves the prediction for the unmagnetized case in $2 < d < 4$ [5, 13]

$$X^\infty = \frac{2}{dD} = \frac{d-2}{d}. \quad (103)$$

Conversely, for $u \gg 1$ one has

$$X^\infty = \frac{d-2}{2} \left[\frac{d+6}{2} A + \left(\frac{d-2}{2} \right)^2 \frac{\Gamma((d+4)/2)}{\Gamma((4-d)/2)\Gamma(d+1)} \frac{d+6}{4} V_d \right]^{-1}, \quad (104)$$

which stated in this form agrees with our earlier result for the fully magnetized case [13]. (The error was in an incorrect expression for V_d ; see appendix A.) In this regime the asymptotic FDR interpolates between $X^\infty = 1/2$ for $d = 2$ (as can be shown by using that $V_d \sim 2/(d-2)$ to leading order [13]) and $X^\infty = 4/5$ for $d = 4$. As is required by continuity with the situation for $d > 4$, the contribution from $C^{(2)}$ vanishes as $d \rightarrow 4$, for any u . An $\epsilon = 4 - d$ -expansion of (101) yields

$$X^\infty(u) = \frac{4(u+1)^4}{3+5(u+1)^4} - 2(u+1)^3 \frac{144+216u+48u^2+160u^3+95u^4+19u^5}{9(3+5(u+1)^4)^2} \epsilon, \quad (105)$$

which in the unmagnetized ($u \ll 1$) and fully magnetized ($u \gg 1$) limits reduces to $X^\infty(u \rightarrow 0) = 1/2 - \epsilon/8$ and $X^\infty(u \rightarrow \infty) = 4/5 - (19/450)\epsilon$, respectively. The former value agrees with the well-known result $X^\infty = (d-2)/d$ for coarsening in the spherical model [5, 13] or the $O(n \rightarrow \infty)$ model [6] from an unmagnetized state. The latter, corrected, value now also agrees with the RG calculations for the longitudinal fluctuations of the $O(n \rightarrow \infty)$ model [14].

One interesting and unexpected feature of (101) and its expansion (105) is that the approach to the large- u limit is non-monotonic: for d close to 4, $X^\infty(u)$ slightly overshoots the limit value ‘plateau’ and then decays down to it, signalling the presence of a weak maximum. Expanding (105) for large u and subtracting off its $u \rightarrow \infty$ asymptote, one sees that the deviation from the plateau is controlled, to leading order, by two terms with opposite signs, scaling respectively as ϵ/u^2 and $-1/u^4$. The maximum occurs where these two terms compete, that is for $u \sim 1/\sqrt{\epsilon}$, or $\bar{u} = u\sqrt{\epsilon} = \mathcal{O}(1)$. Its height above the plateau then scales as ϵ^2 . To get the scaling function determining the shape of the maximum, we therefore normalize the deviation of X^∞ from the large- u plateau by ϵ^2 and define

$$D(\bar{u}) \equiv \lim_{\epsilon \rightarrow 0} \frac{X^\infty(u = \bar{u}/\sqrt{\epsilon}) - \lim_{u \rightarrow \infty} X^\infty(u)}{\epsilon^2} = \frac{2(-18 + 5\bar{u}^2)}{75\bar{u}^4}. \quad (106)$$

This scaling function has its maximum at the finite value $\bar{u} = 6/\sqrt{5}$, as expected, and is positive for $\bar{u} > \sqrt{18/5}$.

Looking next at dimensions further away from $d = 4$, figure 2 shows numerical values of X^∞ for finite u for a few dimensions d between 2 and 4. X^∞ converges to the fully magnetized value (which is near $1/2$ for $d \approx 2$) for large u and to the unmagnetized asymptotic FDR $(d-2)/d$ for $u \rightarrow 0$ as it should. As anticipated, the interpolation between these two limits is not, as in $d > 4$, monotonic: X^∞ initially increases with u but ‘overshoots’ its asymptotic limit. This phenomenon becomes more and more pronounced as $d \rightarrow 2$. For d very close to 2, finally, the maximum turns into two poles in $X^\infty(u)$, with X^∞ being negative in between.

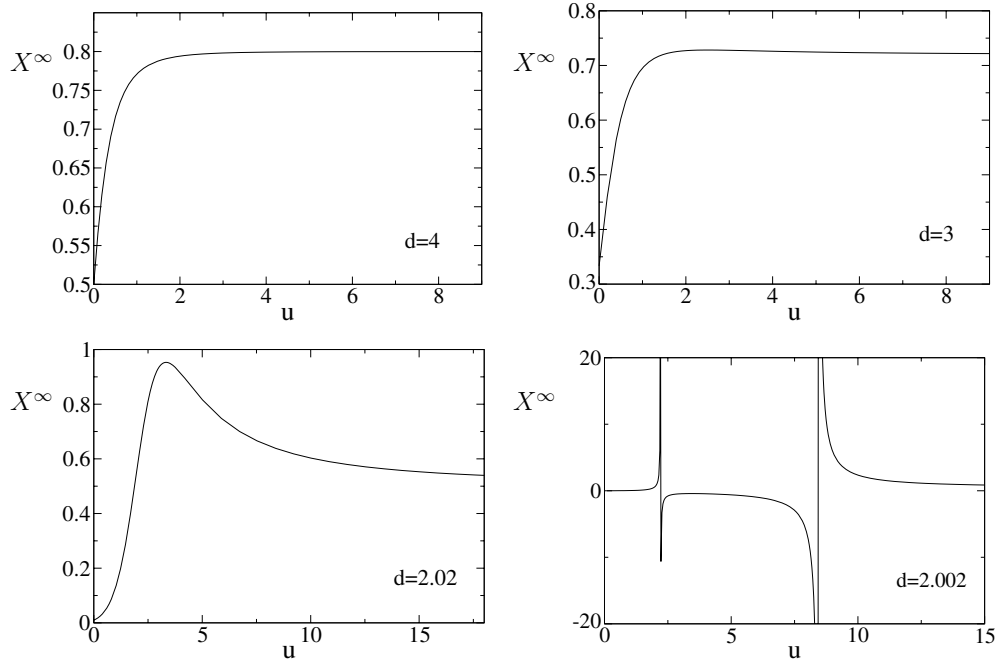


Figure 2. Asymptotic FDR X^∞ for the magnetization versus $u = t_w/\tau_m$, for a few dimensions between 2 and 4 as indicated. As d decreases, $X^\infty(u)$ develops an increasingly pronounced maximum which eventually (see bottom right graph for $d = 2.002$) turns into two poles separated by a region of negative X^∞ .

The first pole is relatively straightforward to analyse from (101). One needs the dependence on $\delta = (d - 2)/2$ of (74)–(77) and (97)–(99) for $\delta \rightarrow 0$. To leading order one finds $A = 3\delta^2/4$, $B = 3\delta/2$, $C = 1$, $D = 1/\delta$. The small δ -limits of V_d , V'_d and V''_d one gets from (97)–(99) by noting that the v -integrals become dominated by their large v tails as $\delta \rightarrow 0$, giving $V_d = 1/\delta$, $V'_d = 2/\delta^2$ and $V''_d = 1/\delta^3$. Gathering these results, equation (101) becomes to leading order

$$X^\infty = (u + 1)^3 \left[(-2u + 1)D + \frac{3}{2}\delta^2 u V''_d \right]^{-1} = \delta \frac{2(u + 1)^3}{2 - u}, \quad (107)$$

which approaches for $u \rightarrow 0$ the unmagnetized limit $X = \delta + \mathcal{O}(\delta^2)$ as it should. For larger u we read off that there is a pole at $u = 2$ beyond which X^∞ is negative. In fact, the expression (107) shows that $X^\infty \approx -2\delta u^2$ for large u whereas we expect convergence to the known limit $X^\infty = 1/2$. The reason is that the limits $u \rightarrow \infty$ and $\delta \rightarrow 0$ do not commute: the approach to the eventual asymptotic value takes place on a scale of values of u that diverges as $\delta \rightarrow 0$.

The form of the response function as given in (58) would suggest that the appropriate diverging u -scale to consider is $u \sim 1/\delta$: in this regime the two terms in the square brackets in (58), which cancel exactly for $u \rightarrow \infty$ and $d \rightarrow 2$, still give a leading order cancellation. However, one finds with a bit of algebra that the limit as $\delta \rightarrow 0$ of X^∞ , taken at fixed $u' = u\delta$, is simply the constant asymptotic value $X^\infty = 1/2$. The crossover to this asymptotic regime must therefore take place on shorter timescales u . To explore this, we need to look more closely at the polynomial structure of X^∞ . As observed, X^∞ can be written as the ratio of fourth-order polynomials,

$$X^\infty = \frac{au^4 + bu^3 + cu^2 + du + e}{a'u^4 + b'u^3 + c'u^2 + d'u + e'} \quad (108)$$

The coefficients can be computed in the limit $\delta \rightarrow 0$ and their leading terms evaluate to

$$\begin{aligned} a &= \delta & a' &= 2\delta \\ b &= 1 + 3\delta & b' &= 2 + \frac{9}{2}\delta \\ c &= 3 + 3\delta & c' &= \frac{11}{2} + \frac{5}{4}\delta \\ d &= 3 + \delta & d' &= -\frac{1}{2\delta} + \frac{9}{4} \\ e &= 1 & e' &= 1 + \frac{1}{\delta}. \end{aligned} \quad (109)$$

We now consider values of u diverging as some generic power of δ , $u = u''\delta^{-\beta}$. In the limit $\delta \rightarrow 0$ a number of terms can then be dropped: e.g. cu^2 in the numerator is always subleading compared to bu^3 because both b and c are order unity but $u \gg 1$ for $\delta \ll 1$. For the same reason the terms proportional to d , e , c' and e' can never be leading. Retaining only the other, potentially leading, terms gives

$$X^\infty = \frac{\delta^{1-4\beta}u''^4 + \delta^{-3\beta}u''^3}{2\delta^{1-4\beta}u''^4 + 2\delta^{-3\beta}u''^3 - \frac{1}{2}\delta^{-1-\beta}u''}. \quad (110)$$

Comparing powers of δ shows that the only values of β for which in the limit $\delta \rightarrow 0$ more than one term survives in either numerator or denominator are $\beta = 1/2, 2/3$ and 1 . The competing terms at $\beta = 2/3$ are both subleading so this case is uninteresting. Only $\beta = 1/2$ therefore remains as a non-trivial exponent value to analyse. One then has explicitly $u'' = u\delta^{1/2}$ and the surviving terms in (110) are

$$X^\infty = \frac{\delta^{-3/2}u''^3}{2\delta^{-3/2}u''^3 - \frac{1}{2}\delta^{-3/2}u''} = \frac{2u''^2}{4u''^2 - 1}. \quad (111)$$

This result matches the magnetized limit $X_m^\infty = 1/2$ for large u'' as it should; for $u'' = 1/2$ it has a pole and for small u'' it is negative and small. In the latter regime, $X^\infty = -2u''^2 = -2\delta u^2$ also matches smoothly with the large- u limit of (107) as it should. Figure 3 demonstrates this behaviour by showing on a logarithmic scale the absolute value of X^∞ versus u . As δ decreases, the second pole moves to larger $u = 1/(2\delta^{1/2})$ as expected while the first one occurs at a finite limiting value of u , $u = 2$.

It would clearly be desirable to understand in more detail the origins of the highly non-trivial behaviour of the asymptotic magnetization FDR for d near 2, and to ascertain how this behaviour is reflected in the corresponding FD plots. The response (58) is always positive, so from the definition of the FDR in (49) singularities in X can arise only from zeros in $\partial_{t_w} C$, i.e. from a non-monotonic dependence of the magnetization correlator on t_w . As we will show, this non-monotonicity arises because the equal-time correlator has a pronounced maximum around $u = t_w/\tau_m = 1$, and this large variance of the magnetization fluctuations leaves its imprint in the two-time correlator as a weak maximum.

The main difficulty we now face is to obtain the behaviour of the correlator also for finite x rather than just $x \gg 1$. This is made possible by the following observation: in the $\delta = (d - 2)/2 \rightarrow 0$ -limit, the scaling function $\mathcal{F}_M(x/y, uy)$ from (54) develops a non-integrable singularity at $y = x$. This concentrates the weight of any integrand into this region, so that for any function $f(y, u)$ which is smooth at $y = x$

$$\int_0^x dy \mathcal{F}_M\left(\frac{x}{y}, uy\right) f(y, u) \rightarrow \frac{ux^{1/2}}{ux + 1} f(x, u) \quad (112)$$

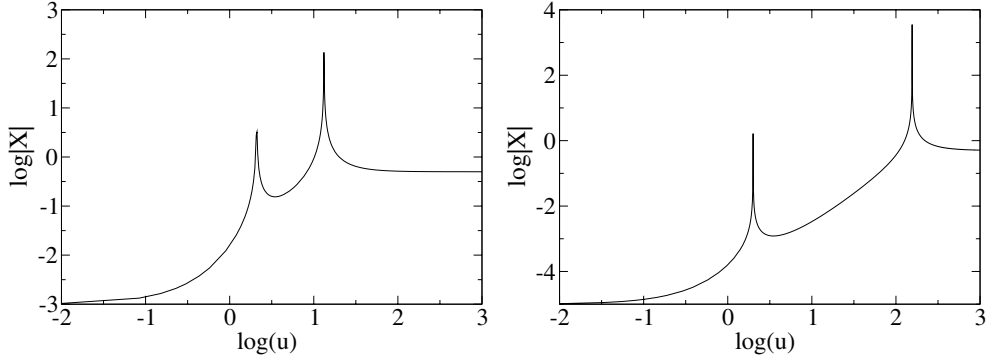


Figure 3. Log–log plot of the absolute value of the asymptotic FDR X^∞ versus u , for $\delta = 10^{-3}$ (left) and $\delta = 10^{-5}$ (right). Note that the second pole moves to the right as δ decreases, with the expected proportionality $u \sim \delta^{-1/2}$. Here and in the following plots, $\log \equiv \log_{10}$.

as $\delta \rightarrow 0$, i.e. $\mathcal{F}_M(x/y, uy)$ acts effectively as $\mathcal{F}_M(x/y, uy) = [ux^{1/2}/(ux + 1)]\delta(x - y)$. The same observation applies to $\mathcal{F}_M(1/y_w, uy_w)$, which is obtained by setting $x = 1$ and $y = y_w$. These approximations will yield the leading terms in the correlator in the limit $\delta \rightarrow 0$. We will also use them for dimensions slightly above 2 to explore numerically the x -dependence of the correlator and the resulting FD behaviour. Even though the results here no longer have the character of a systematic expansion in δ , they will give insights into the non-trivial d -dependence of the FD plots for d close to 2.

Using the approximation (112), the contributions to $C^{(1)}$ coming from S vanish because the first argument of $\mathcal{F}_M(x/y, uy)$ is always $\geq x > 1$. The remaining term F is given explicitly in (66) and taking the $\delta \rightarrow 0$ limit gives

$$\frac{C^{(1)}(t, t_w)}{T_c t} = \frac{2x^{-1/2}}{\delta(ux + 1)^{3/2}(u + 1)^{3/2}}(1 + u\delta + u^2\delta^2). \tag{113}$$

We note that a naive application of the δ -approximation explained above would in this case give an incorrect result, because it produces a leading order cancellation of terms when $u \sim 1/\delta$. The remaining subleading term is then of the same order as the first correction to the δ -approximation. One can nevertheless check that the second contribution, S , to $C^{(1)}$ always remains negligible compared to F because it is subject to a similar cancellation.

Next we want to compute $C^{(2)}$. In (90) the first integral again vanishes because the first argument of \mathcal{F}_M is always > 1 , so

$$W = \frac{T_c^2 \sigma_d}{\mu_d} \left(\frac{x}{y_w}\right)^{(4-d)/4} \frac{x^{3-d/2}}{(uy_w + 1)(ux + 1)^{3/2}} \int_0^1 dz dz' (zz')^{(d-4)/2} \times \left(\frac{x}{y_w} - z - z' + 1\right)^{-1} (uy_w z + 1)(uy_w z' + 1), \tag{114}$$

which for small δ evaluates to

$$W = \frac{T_c^2 \sigma_d}{\mu_d} \left(\frac{x}{y_w}\right)^{(4-d)/4} \frac{x^{3-d/2}}{(uy_w + 1)(ux + 1)^{3/2}} \left\{ u^2 y_w^2 \left[\frac{x}{y_w} \ln \left(1 - \frac{y_w^2}{x^2} \right) + \ln \left(\frac{x + y_w}{x - y_w} \right) \right] + \frac{2uy_w}{\delta} \ln \left(\frac{x + y_w}{x} \right) + \frac{1}{\delta^2} \frac{y_w}{x + y_w} \right\}. \tag{115}$$

The y_w -integral in (87) can be performed by again using the δ -approximation and one finally gets

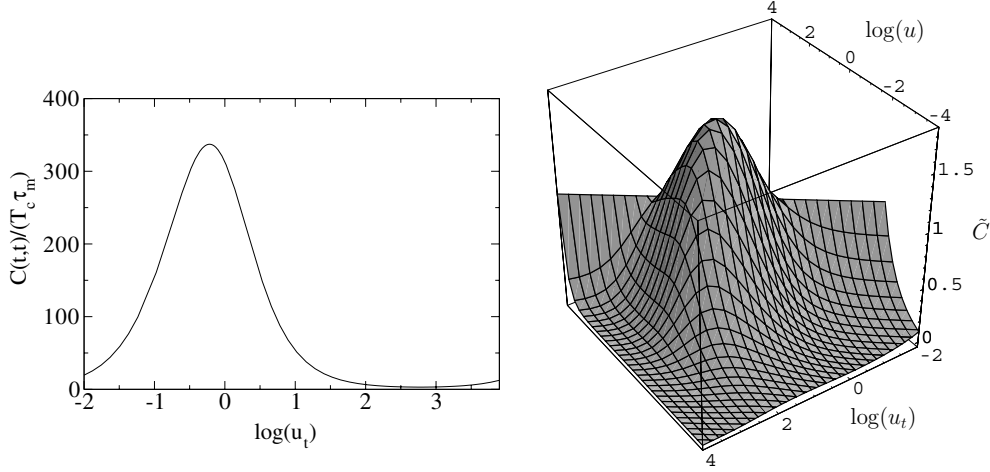


Figure 4. Left: the equal-time correlator $C(t, t)/(T_c \tau_m)$ versus $\log(u_t)$ for $\delta = 10^{-3}$ shows pronounced non-monotonic behaviour. Right: normalized correlator \tilde{C} for the same $\delta = 10^{-3}$, plotted versus $\log(u)$ and $\log(u_t)$. Note the non-monotonicities in the u -dependence around $\log(u) = 0$.

$$\frac{C^{(2)}(t, t_w)}{T_c t} = \frac{ux^{1/2}}{\delta(u+1)^{3/2}(ux+1)^{3/2}} \left\{ \frac{1}{x+1} + 2u\delta \ln\left(\frac{x+1}{x}\right) + u^2\delta^2 \left[x \ln\left(1 - \frac{1}{x^2}\right) + \ln\left(\frac{x+1}{x-1}\right) \right] \right\}. \quad (116)$$

Adding the results (113) and (116) gives the magnetization correlation function for generic x and d close to 2. By pulling a factor of x into the curly bracket of the latter, one sees that the resulting terms in the bracket are of $\mathcal{O}(1)$, $\mathcal{O}(\delta u)$ and $\mathcal{O}(\delta^2 u^2)$ for all x , as in (113). But the prefactor in (116) is larger by a factor of u , so for $u \gg 1$ we can neglect $C^{(1)}$ against $C^{(2)}$. This implies that we can always drop the $\mathcal{O}(\delta u)$ and $\mathcal{O}(\delta^2 u^2)$ terms in $C^{(1)}$: either $u = \mathcal{O}(1)$, and then they are subleading compared to the $\mathcal{O}(1)$ term in $C^{(1)}$, or $u \gg 1$ and they are small compared to the corresponding terms in $C^{(2)}$. Changing then also to a normalization with $1/\tau_m$ instead of $1/t$, we can express the leading order terms for $\delta \rightarrow 0$ of the magnetization correlator as a function of the scaled times $u = t_w/\tau_m$ and $u_t = t/\tau_m = xu$ in the form

$$\frac{C(t, t_w)}{T_c \tau_m} = \frac{u^{1/2}u_t^{3/2}}{\delta(u+1)^{3/2}(u_t+1)^{3/2}} \left\{ \frac{2}{u_t} + \frac{u}{u+u_t} + 2u\delta \ln\left(\frac{u+u_t}{u_t}\right) + u^2\delta^2 \left[\frac{u_t}{u} \ln\left(1 - \frac{u^2}{u_t^2}\right) + \ln\left(\frac{u+u_t}{u_t-u}\right) \right] \right\}. \quad (117)$$

The equal-time correlator is then obtained by taking the limit $u \rightarrow u_t$,

$$\frac{C(t, t)}{T_c \tau_m} = \frac{u_t(4+u_t+4(\ln 2)\delta u_t^2+4(\ln 2)\delta^2 u_t^3)}{2\delta(u_t+1)^3}. \quad (118)$$

Evaluating this numerically for small δ as shown in figure 4, we see that it is non-monotonic in u_t as anticipated. In fact, the expression (118) shows directly that the height of the peak at $u_t = \mathcal{O}(1)$ diverges as $1/\delta$ for $\delta \rightarrow 0$, whereas for $u_t = \mathcal{O}(1/\delta)$ the result is of order unity. On the right of figure 4 we demonstrate that the peak in the equal-time correlation function does indeed cause corresponding non-monotonic behaviour in the (normalized) two-time correlator

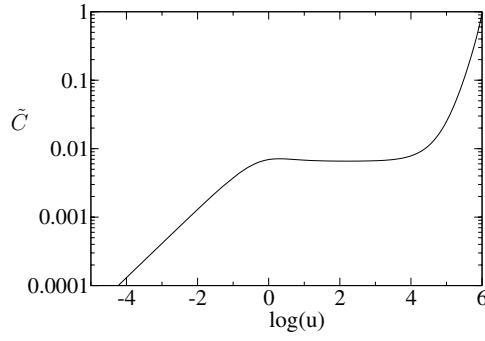


Figure 5. Normalized correlator versus $\log(u)$ at fixed $u_t = 10^6$ and $\delta = 10^{-5}$.

$\tilde{C}(t, t_w) = C(t, t_w)/C(t, t)$ in the region where $u = \mathcal{O}(1)$. Note that in \tilde{C} the prefactors $T_c \tau_m$ that we have isolated on the left of the expressions above cancel and we obtain a function of only u_t and u ,

$$\tilde{C}(u_t, u) = \left(\frac{u_t + 1}{u + 1}\right)^{3/2} \frac{2u^{1/2}u_t^{1/2}}{4 + u_t + 4(\ln 2)\delta u_t^2 + 4(\ln 2)\delta^2 u_t^3} \left\{ \frac{2}{u_t} + \frac{u}{u + u_t} + 2u\delta \ln\left(\frac{u + u_t}{u_t}\right) + u^2\delta^2 \left[\frac{u_t}{u} \ln\left(1 - \frac{u^2}{u_t^2}\right) + \ln\left(\frac{u + u_t}{u_t - u}\right) \right] \right\}. \quad (119)$$

We now analyse more closely the nature and scaling of the non-monotonocities of the normalized two-time correlator. For u_t of order unity, all terms involving powers of δ can be dropped in (119). The resulting function is monotonically increasing in $u = t_w/\tau_m = 0 \dots u_t$ for $u_t > \sqrt{13} - 3 \approx 0.61$. For larger u_t it has a maximum in u whose position shifts from $\sqrt{13} - 3$ to an asymptotic limit of 2 as u_t increases. If one keeps the terms that are subleading in δ , one sees that \tilde{C} contains a contribution scaling as $\delta^2(u_t - u) \ln(u_t - u)$. This yields a term $-\delta^2 \ln(u_t - u)$ in the u -derivative of \tilde{C} which diverges to $+\infty$ as $u \rightarrow u_t$, and so gives a positive sign for the derivative in the limit. Whenever \tilde{C} has a maximum as a function of u it therefore also has an associated minimum, but this is essentially undetectable as it occurs extremely close to u_t , for $u_t - u \sim \exp(-\text{const}/\delta^2)$.

Moving to larger u_t of order $1/\delta$, the position of the minimum in \tilde{C} becomes clearly separate from u_t . An example of this is shown in figure 5, which graphs the normalized correlator as a function of $\log(u)$ for a fixed value of u_t with $u_t\delta = 10$: one discerns a small maximum followed by a broad minimum. (Numerically, one finds that these features merge once d gets sufficiently far above 2, restoring monotonicity.) The maximum in u is, for $u_t \sim 1/\delta$, always located at $u = 2$; this matches the behaviour discussed above for large u_t of $\mathcal{O}(1)$. Explicitly, if we let δ tend to 0 in the normalized correlator at fixed u and $u'_t = u_t\delta$ we get

$$B(u'_t, u) \equiv \lim_{\delta \rightarrow 0} \tilde{C}(u'_t/\delta, u) = \frac{2\sqrt{u}(u + 2)}{(u + 1)^{3/2}(1 + 4(\ln 2)u'_t + 4(\ln 2)u'^2_t)}. \quad (120)$$

The result is shown in figure 6 (left) and does have a maximum at $u = 2$ as anticipated. This value makes sense since it was also the point where the asymptotic FDR $X^\infty(u)$ diverges for d close to 2. (Given that $u_t \sim 1/\delta$ we are automatically in the asymptotic regime $u_t \gg u$.)

The position u_{\min} of the corresponding minimum of \tilde{C} as a function of u is somewhat more subtle. For $u'_t < 1/2$, it is located at $u_{\min} \sim 1/\delta$, i.e. $u_{\min} < u_t$ but with the two values

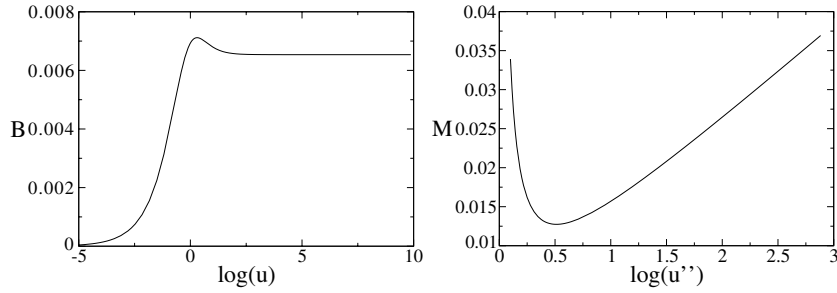


Figure 6. Position and scaling of the maximum and the minimum of the normalized two-time correlator as a function of u or $u'' = u\delta^{1/2}$ for a fixed value of $u_t \sim 1/\delta$; for the example in the plot we have taken $u_t\delta = u'_t = 10$.

being of the same order. As $u'_t \rightarrow 1/2$ from below, $u_{\min}\delta \rightarrow 0$; for even larger values of u'_t , one finds a different scaling $u_{\min} \sim \delta^{-1/2}$ so that always $u_{\min} \ll u_t$. To find the minimum position in this regime we need to fix $u'' = u\delta^{1/2}$ in the normalized correlator when letting $\delta \rightarrow 0$. The typical values of \tilde{C} in this regime turn out to be only $\mathcal{O}(\delta^{1/2})$ above the plateau $B(u'_t, u \rightarrow \infty)$ so we subtract off the latter and divide by $\delta^{1/2}$ to define

$$M(u'_t, u'') = \lim_{\delta \rightarrow 0} \frac{\tilde{C}(u'_t/\delta, u''/\delta^{1/2}) - B(u'_t, u \rightarrow \infty)}{\delta^{1/2}} \tag{121}$$

$$= \frac{u'_t + u''^2(-2 + 4u'_t)}{u''u'_t(1 + 4(\ln 2)u'_t + 4(\ln 2)u_t'^2)}. \tag{122}$$

We show a sample plot of this, for a specific value of $u'_t > 1/2$, in figure 6 (right). The minimum of M occurs at $u''_{\min} = [u'_t/(4u'_t - 2)]^{1/2}$, which for large u'_t yields $u''_{\min} = 1/2$. This matches the position of the second divergence of the asymptotic FDR $X^\infty(u)$, as it should.

As explained above, the derivative of the two-time correlator is always dominated by a logarithmically divergent term in the limit $u \rightarrow u_t$; including prefactors, this reads $\partial_{t_w} C(t, t_w) = -T_c\delta \ln(u'_t - u')$ in the regime $u_t = u'_t/\delta, u = u'/\delta$. The response function (58) is then dominated by the same logarithmic terms,

$$R(u'_t, u') = \frac{\delta}{u'_t} \left[1 - u'_t \ln \left(1 - \frac{u'}{u'_t} \right) \right] \approx -\delta \ln(u'_t - u'). \tag{123}$$

The last approximation, which gives the dominant term for $u' \rightarrow u'_t$, shows that the initial (negative) slope of the FD plot is always exactly equal to 1. However, as u'_t becomes small this becomes undetectable because so does the logarithmic singularity in the correlator.

We are now in a position to analyse the magnetization FD plots for d near 2. We consider normalized plots ($\tilde{\chi}$ versus \tilde{C}) as in $d > 4$, holding u_t fixed for each plot as before to get a valid connection with the FDR X , and varying u . We obtain $\tilde{\chi}$ by numerical integration of (58), according to definition (50), and then dividing by (118). The asymptotic FDR $X^\infty(u)$ that we have calculated applies in the limit $u_t \gg u$, corresponding to the region in the top left-hand corner of an FD plot. Starting from the top left corner ($u \rightarrow 0$) we then expect to see in the FD plots the slopes varying as given by $X^\infty(u)$: initially small (of $\mathcal{O}(\delta)$) and negative as usual, then turning positive and of order unity, and finally negative again. This S-shape should be present for large u_t ; for smaller u_t , only part of this variation will be accessible because $u \leq u_t$.

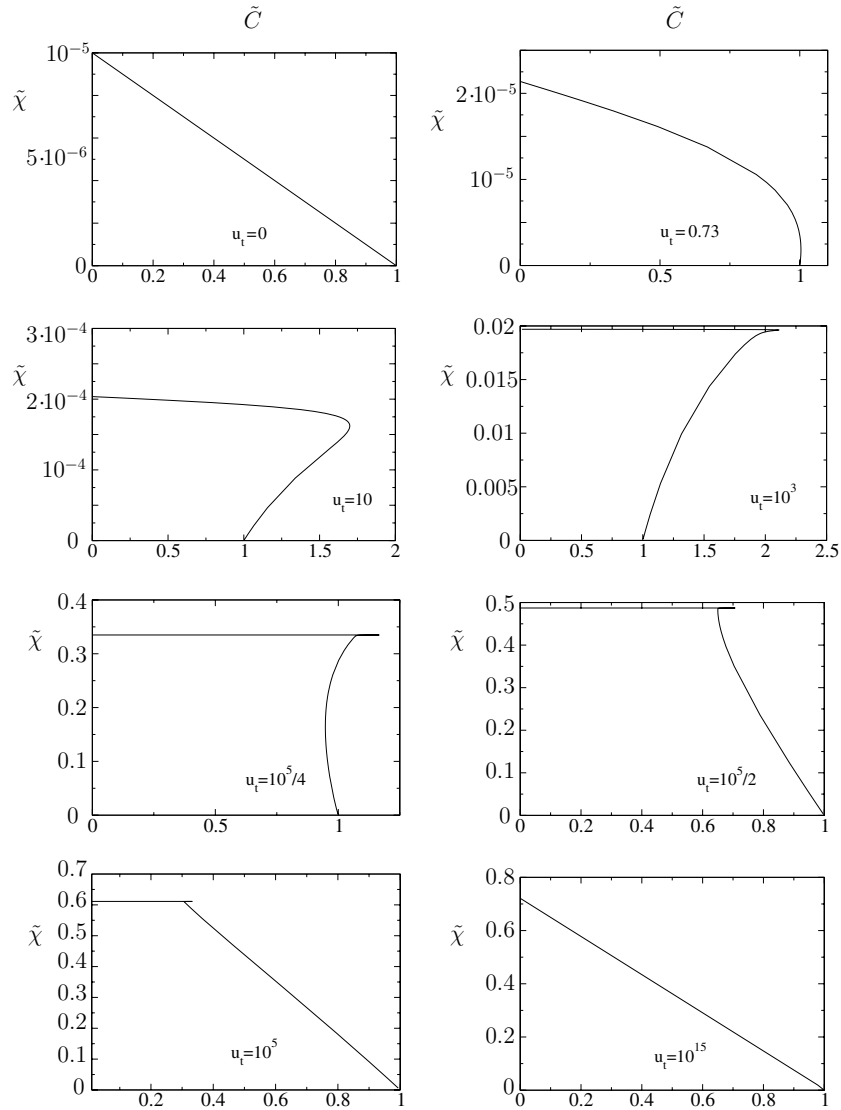


Figure 7. Normalized FD plots for d close to 2, showing normalized susceptibility $\tilde{\chi}$ versus normalized correlation \tilde{C} , for $\delta = 10^{-5}$ and increasing values of u_t as shown in each plot. Once the S-shape appears, it remains present for all larger u_t but gets squashed into a region in the top left-hand corner scaling as $1/(u_t\delta)^2$. The slope where the plot meets the y -axis is always given by $X^\infty(u = 0) \approx \delta$.

For d close to 2 the above expectations are indeed borne out by numerical evaluation as illustrated in figure 7 for $\delta = 10^{-5}$. As u_t increases, we start from the fully linear FD plot of the unmagnetized case, with negative slope $X = X^\infty(u = 0) \approx \delta$. A section of much larger X then grows and eventually ‘flips’ to the right, producing a region of negative FDRs. For much larger values ($u_t \sim \delta^{-1}$) the beginning of the FD plot (equal times, where the plot meets the horizontal axis) eventually swings back to the left to return to the conventional negative slope. The initial slope of -1 also becomes visible. At this stage the expected S-shape is complete;

it then shrinks progressively towards the top left corner as u_t grows and the rest of the plot approaches the close-to-linear shape [13] for the fully magnetized case. The region of the plot occupied by the ‘S’ scales as $1/u_t^2 = 1/(u_t\delta)^2$ for $u_t \gg 1/\delta$. This is clear from (120) which gives the typical values of \tilde{C} at the maximum and minimum, i.e. at the right and left boundary of the ‘S’. The S-shaped region ends where the plot meets the y -axis with an asymptotic slope that is u_t -independent: this point corresponds to $u \rightarrow 0$, so we are always in the regime $u_t \gg u$ where the asymptotic FDR X^∞ applies and the negative slope is $X^\infty(u=0) \approx \delta$. (In fact, this argument applies for any u_t , whether or not an actual S-shape is present.)

The crossover between unmagnetized and fully magnetized behaviour can also be seen from the u_t -dependence of the y -axis intercept of the FD plot, which can be thought of as its ‘axis ratio’ Y . This is found from the large- x limit at fixed u_t of the susceptibility, multiplied by T_c and normalized by the equal-time correlator $C(t, t)$ from (118). The former is determined from the response function (58) by integration, $\chi(t, t_w) = \int_{t_w}^t dt' R(t, t')$. Rescaling $t' = zt$ gives

$$\chi(t, t_w) = \frac{t}{(u_t + 1)^{1/2}} \int_{1/x}^1 dz z^{(d-4)/4} (u_t z + 1)^{1/2} \left[1 - \frac{u_t}{u_t + 1} (1 - z)^{(d-2)/2} \right]. \quad (124)$$

In the small δ -limit at fixed $u_t = \mathcal{O}(1)$, the square bracket simplifies to $1/(u_t + 1)$ and the z -integral can be done explicitly. Multiplying by $T_c/C(t, t)$ gives for the axis ratio in this regime

$$Y = \delta \frac{2(u_t + 1)^{3/2}}{4 + u_t} [\sqrt{1 + u_t} + u_t^{-1/2} \ln(\sqrt{u_t} + \sqrt{1 + u_t})]. \quad (125)$$

This is of order δ as expected from the FD plots in figure 7. For $u_t \rightarrow 0$ one gets $Y = \delta$ exactly, consistent with the known results for the unmagnetized case [13]; for large u_t , on the other hand, $Y = 2\delta u_t$.

In the regime $u_t \sim 1/\delta$ one finds similarly, by setting $u_t = u'_t/\delta$ and taking $\delta \rightarrow 0$

$$Y = \frac{2u'_t(1 + u'_t)}{1 + 4(\ln 2)u'_t + 4(\ln 2)u'^2_t}. \quad (126)$$

This is of order unity, again consistent with the FD plots shown above. For $u'_t \ll 1$ it approaches $2u'_t$, matching the result from the previous regime, while for $u'_t \rightarrow \infty$ one retrieves $Y = 1/(2 \ln 2)$ in agreement with the result for the fully magnetized case [13]. We show the two scaling functions together in figure 8, for the example $\delta = 10^{-3}$. As expected the two functions agree in the intermediate regime $1 \ll u_t \ll 1/\delta$, where the axis ratio crosses over from values typical of unmagnetized coarsening ($Y \sim \delta$) to the values of order unity for the magnetized scenario.

5. Discussion

We have used exact calculations to study the crossover from unmagnetized to magnetized initial conditions in the critical coarsening of the spherical ferromagnet. Our focus was on the correlation and response functions of the overall magnetization and the associated non-equilibrium fluctuation–dissipation (FD) relations. We derived, in particular, the first exact results (in the non-trivial regime $d < 4$) for the crossover function $X^\infty(u)$ governing the behaviour of the asymptotic FD ratio X^∞ ; $u = t_w/\tau_m$ is the appropriate scaling variable, namely the ratio of the earlier measurement time t_w and the timescale $\tau_m \sim 1/m_0^2$ set by the initial magnetization m_0 . While $X^\infty(u)$ does interpolate between the known unmagnetized ($u \ll 1$) and fully magnetized ($u \gg 1$) limits, we found that unexpectedly the behaviour for

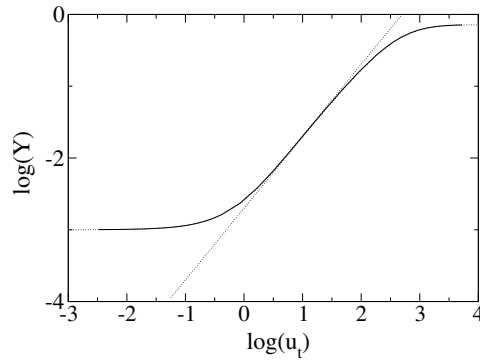


Figure 8. Axis ratio Y of FD plot, given by the asymptotic normalized susceptibility $\bar{\chi}(u_t, u \rightarrow 0)$, versus $\log(u_t)$ for $\delta = 10^{-3}$. The scaling functions (125) and (126) are shown and match in the crossover regime ($1 \ll u_t \ll 1/\delta$) as expected. The dotted lines represent their continuations towards larger and smaller u_t , respectively. For $u_t = \mathcal{O}(1)$, Y is $\mathcal{O}(\delta)$ as expected from the unmagnetized case, whereas for $u_t \gg 1/\delta$ we retrieve the magnetized limit $Y = 1/(2 \ln 2)$.

intermediate u is not monotonic. In fact, for dimensions $d \approx 2$ close to the lower critical dimension these non-monotonicities turn into pole singularities in $X^\infty(u)$.

We traced this unusual behaviour to a non-monotonic dependence on the earlier time t_w of the two-time magnetization correlator $C(t, t_w)$, which displays a weak maximum at $u = \mathcal{O}(1)$ and a corresponding minimum at $u = \mathcal{O}(\delta^{-1/2})$ (for sufficiently large $u_t = t/\tau_m = \mathcal{O}(\delta^{-1})$). We interpreted the maximum as the result of an unusually large variance of the magnetization fluctuations in this region, corresponding to a strong peak in the *equal-time correlator* $C(t_w, t_w)$. The maximum and minimum of $C(t, t_w)$ also manifest themselves as S-shapes in the magnetization FD plots for $d \approx 2$.

As an aside, we note that non-monotonicities in the asymptotic FDR have previously been observed as a function of the *lengthscale* being probed [14]. Also here the effect gets stronger as $d \rightarrow 2$. On the other hand, the non-monotonic dependence on the lengthscale disappears for large enough u at any $d > 2$, so that it is unclear whether the physical mechanism at work here is related to that causing the complicated dependence of X^∞ on scaled system age that we saw above.

The calculations for the more general crossover case revealed a typographical error in our earlier study of fully magnetized initial conditions [13]. Having corrected this, the expansion to first order in $4 - d$ of the asymptotic FDR X^∞ for the magnetized case now *agrees* with the result of an RG calculation for the $O(n \rightarrow \infty)$ model [14]. This agreement suggests, non-trivially, that the spherical and $O(n \rightarrow \infty)$ models are closely related even beyond the leading order Gaussian description of their dynamics. One might then suspect similar agreement also with the n -vector model; to verify this, it would be desirable to extend existing RG expansions around $d = 2$ [18] beyond the leading term $X^\infty = 1/2 + \mathcal{O}(d - 2)$. Note that in comparing the spherical with the $O(n)$ and n -vector models one has to look at the *longitudinal* degrees of freedom in the latter since these are those which—like the magnetization in the spherical case—have nonzero average. The *transverse* fluctuations in coarsening from a state with finite initial magnetization behave differently, giving in the $O(n \rightarrow \infty)$ model an asymptotic FDR of $X^\infty = d/(d + 2)$ [15]. One expects that the transverse fluctuations in the n -vector model would give the same FDR for $n \rightarrow \infty$. This is consistent with the first-order expansion $X^\infty = 1/2 + (d - 2)/8$ calculated in [18]. Intriguingly, even though the spherical model with

its single degree of freedom per lattice site has no direct analogue of transverse fluctuations, it gives the same FDR $X^\infty = d/(d + 2)$ for short-range observables when the system coarsens from an initially magnetized state.

In future work, an issue of obvious interest would be to understand how generic our results are, i.e. whether similar non-monotonicitics appear also in true short-range models. Field-theoretic calculations near $d = 4$ [10], for e.g., the $O(n)$ model should in principle be possible, and could be directly compared to the expansion (105) of our results near $d = 4$. Our analysis also suggests that if similar expansions were carried out near $d = 2$ [18], very rich behaviour could result.

Appendix A. Corrections to [13]

In this appendix, we list the required corrections to the relevant equations of [13]. The source of the error was equation (8.94) of [13]: it should be replaced by

$$\frac{CC(t, t_w)}{CC(t, t)} = \mathcal{G}(t/t_w), \quad \mathcal{G}(x) = \begin{cases} \frac{\int dw w^{(d-6)/2} \mathcal{F}_C^2(w) e^{-2(x-1)w}}{x \int dw w^{(d-6)/2} \mathcal{F}_C^2(w)} & \text{for } x \geq 1 \\ x^{(d-4)/2} \mathcal{G}(1/x) & \text{for } x \leq 1 \end{cases}. \quad (\text{A.1})$$

The old version had an erroneous $x^{(d-6)/2}$ rather than $x^{(d-4)/2}$ in the second line of the curly bracket. All other mistakes are due to trivial propagation of the one above. This affects the first integral in each of equations (8.99) and (8.100), whose correct versions are

$$\frac{2\mu_d U}{\tilde{\gamma}_d} = \int_0^1 du \mathcal{F}_M(x/uy_w) u^{(d-2)/4} \mathcal{G}(1/u) + \int_1^{x/y_w} du \mathcal{F}_M(x/uy_w) u^{(6-d)/4} \mathcal{G}(u) \quad (\text{A.2})$$

$$= \int_1^\infty du \mathcal{F}_M(xu/y_w) u^{-(d+6)/4} \mathcal{G}(u) + \int_1^{x/y_w} du \mathcal{F}_M(x/uy_w) u^{(6-d)/4} \mathcal{G}(u). \quad (\text{A.3})$$

This leads to

$$U = \frac{T_c}{\Gamma(\frac{d-2}{2})\Gamma(\frac{4-d}{2})} \left[\int_1^\infty du \mathcal{F}_M\left(\frac{xu}{y_w}\right) u^{-(d+10)/4} \right. \\ \times \int_0^1 dy \int_0^1 dy' (yy')^{(d-2)/2} (1 - y - y' + u)^{-d/2} \\ \left. + \int_1^{x/y_w} du \mathcal{F}_M\left(\frac{x}{uy_w}\right) u^{(2-d)/4} \int_0^1 dy \int_0^1 dy' \dots \right] \quad (\text{A.4})$$

in place of equation (8.104). Equations (8.105), (8.107), (8.108), (8.113) as written in terms of V_d are correct, but V_d itself as stated in (8.106) is incorrect. The correct version is (97) in the main text. The limiting value V_4 for $d \rightarrow 4$ can be worked out explicitly as $V_4 = 5/6$ and must replace equation (8.115). This enters X^∞ (denoted X_m^∞ in [13]) only at the first order in an $\epsilon = 4 - d$ -expansion

$$X^\infty = \frac{4}{5} - \frac{19}{450}\epsilon + \mathcal{O}(\epsilon^2) \quad (\text{A.5})$$

which needs to replace equation (8.116).

In the opposite limit $d \rightarrow 2$, the error affects again only subleading contributions, so, e.g., equation (8.117) stands as written. The first correction a_0 in the $\delta = (d - 2)/2$ -expansion of $V_d = 1/\delta + a_0 + \dots$ can be obtained as $a_0 = -3/2$ (rather than $a_0 = -1/2 - \pi^2/12$ [13]) following the reasoning in [13]. The correct expansion of X^∞ near $d = 2$ then becomes

$$X^\infty = \frac{1}{2} + \frac{5}{16}(d - 2) + \mathcal{O}((d - 2)^2) \quad (\text{A.6})$$

which should replace equation (8.118).

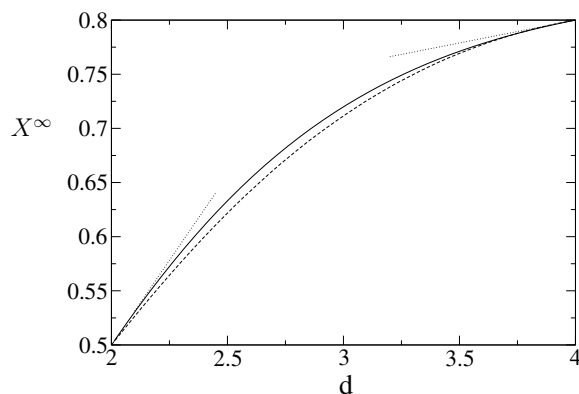


Figure A1. Asymptotic FDR X^∞ for the magnetization, for critical coarsening with nonzero initial magnetization. Dashed line: old, incorrect version from [13]; solid line: correct version; dotted lines indicate the (corrected) first-order expansions (A.5) and (A.6) near $d = 4$ and $d = 2$, respectively.

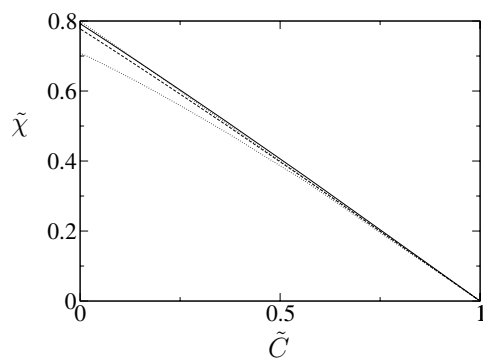


Figure A2. Normalized magnetization FD plot showing normalized susceptibility $\tilde{\chi}$ versus normalized correlation \tilde{C} in the limit of long times. For $d = 2$ (lower dotted line) and $d = 4$ (upper dotted line) the old and the corrected versions coincide. The deviations between the two versions (correct: full line, old: dashed line) are largest in $d = 3$. The correct plot here is somewhat closer to the straight line obtained for $d = 4$, lying very slightly above it in the right-hand part of the plot.

Figure A1 shows the correct d -dependence of X^∞ compared to the erroneous version from [13]. As expected from the discussion above, the quantitative corrections are largest around $d = 3$ and vanish as d approaches 2 or 4. For the sake of comparison, we also show in figure A2 the corrected magnetization FD plots: in $d = 2$ and $d = 4$ these are as before, whereas for $d = 3$ small quantitative differences are just about visible.

Appendix B. Comparison of $L^{(2)}$ with previous results

We show briefly in this appendix that the general and exact solution for $L^{(2)}$

$$L^{(2)}(t, t_w) = -\partial_t N(t, t_w) = \frac{g(t_w)}{g(t)} L_{\text{eq}}^{(2)}(t - t_w) + \frac{g'(t)g(t_w)}{g^2(t)} N_{\text{eq}}(t - t_w) \tag{B.1}$$

reproduces the long-time results obtained for the unmagnetized and fully magnetized limits in [13]. Beginning with $d < 4$, because $g'(t)/g(t) \sim 1/t$ the second term on the rhs is non-negligible only in the aging regime ($t/t_w = x > 1$, hence $t - t_w \sim t_w \gg 1$) where $N_{\text{eq}}(t - t_w) = [2(t - t_w)/(4 - d)]L_{\text{eq}}^{(2)}(t - t_w)$ from (40). Inserting (16) then gives

$$L^{(2)}(t, t_w) = L_{\text{eq}}^{(2)}(t - t_w)x^\kappa \left[\frac{u + 1}{ux + 1} + \frac{(\alpha ux - \kappa)(u + 1)}{(ux + 1)^2} \frac{2}{4 - d} \frac{x - 1}{x} \right]. \quad (\text{B.2})$$

This result is of the general scaling form $L^{(2)}(t, t_w) = L_{\text{eq}}^{(2)}(t - t_w)\mathcal{F}_L(x, u)$, with the scaling function \mathcal{F}_L providing a multiplicative aging correction of the equilibrium result. For $u \ll 1$ or $u \gg 1$ the u -dependence drops out and we obtain the scaling functions found previously for $d < 4$. Specifically, for $u \ll 1$ the square bracket simplifies to $1/x$ and one gets the unmagnetized result $\mathcal{F}_L(x) = x^{(2-d)/2}$ as in [13], whereas for $u \gg 1$ one retrieves the expression for the fully magnetized case, $\mathcal{F}_L(x) = x^{(2-d)/2}[2 + (2 - d)x]/(4 - d)$, also derived in [13].

For $d > 4$ the situation is a little more complicated because for long times, from (40),

$$N_{\text{eq}}(t - t_w) = 1/\mu_d + [2(t - t_w)/(d - 4)]L_{\text{eq}}^{(2)}(t - t_w) \quad (\text{B.3})$$

has a constant part of order unity. This means that, e.g. in the unmagnetized case, where $g(t)$ approaches a constant and one would normally drop the term proportional to $g'(t)$ in (43), a subleading contribution needs to be retained in $g'(t)$. The form of this can be found from the Laplace transform (9) together with (41): $g'(t)$ then has the transform

$$s\hat{g}(s) - g(0) = [s + 2T_c - \hat{L}_{\text{eq}}^{(2)}(0) - (\hat{L}_{\text{eq}}^{(2)}(s) - \hat{L}_{\text{eq}}^{(2)}(0))] \times \left[\frac{m_0^2}{s} + (1 - m_0^2)\hat{f}(0) + (1 - m_0^2)(\hat{f}(s) - \hat{f}(0)) \right]. \quad (\text{B.4})$$

Transforming to the time domain gives

$$g'(t) = m_0^2(2T_c - \hat{L}_{\text{eq}}^{(2)}(0)) + m_0^2 \int_t^\infty dt' L_{\text{eq}}^{(2)}(t') - (1 - m_0^2)\hat{f}(0)L_{\text{eq}}^{(2)}(t). \quad (\text{B.5})$$

Here we have neglected the terms arising from $\hat{f}(s) - \hat{f}(0)$, which decay as $t^{-d/2}$ or even faster at long times and so will be irrelevant below. We can now systematically analyse the order of the various contributions to (B.1) in the long-time limit, obtained by fixing $u = t_w/\tau_m$ and $x = t/t_w$ and taking $t_w \rightarrow \infty$. In the second term of (B.1), $N_{\text{eq}}(t - t_w)$ scales as $\mathcal{O}(t_w^0) + \mathcal{O}(t_w^{(4-d)/2})$ from (B.3). For $g'(t)$, we note that in $d > 4$ the first bracket in (B.5) is equal to μ_d^{-1} . Using also $m_0^2 = c/\tau_m = cu/t_w$ and $\hat{f}(0) = c$ gives for long times

$$g'(t) = \frac{c}{\mu_d} \left[\frac{u}{t_w} + \mu_d \left(\frac{2ux}{d - 4} - 1 \right) L_{\text{eq}}^{(2)}(t) \right], \quad (\text{B.6})$$

i.e. $g'(t) = \mathcal{O}(ut_w^{-1}) + \mathcal{O}(t_w^{(2-d)/2})$. Integrating w.r.t. t yields $g(t) = \mathcal{O}(t_w^0) + \mathcal{O}(t_w^{(4-d)/2})$; the leading order term is given explicitly in (13). Finally, we have $L_{\text{eq}}^{(2)}(t - t_w) \sim (t - t_w)^{(2-d)/2} = t_w^{(2-d)/2}(x - 1)^{(2-d)/2} = \mathcal{O}(t_w^{(2-d)/2})$. One now sees that the unmagnetized case $u = 0$ is special: both terms of (B.1) then scale as $\mathcal{O}(t_w^{(2-d)/2})$, with $N_{\text{eq}}(t - t_w) = 1/\mu_d$, $g(t) = g(t_w) = c/\mu_d$, $g'(t) = -cL_{\text{eq}}^{(2)}(t)$ to leading order so that

$$L_{\text{eq}}^{(2)}(t, t_w) = L_{\text{eq}}^{(2)}(t - t_w) - L_{\text{eq}}^{(2)}(t) = L_{\text{eq}}^{(2)}(t - t_w) \left[1 - \left(\frac{x - 1}{x} \right)^{(d-2)/2} \right]. \quad (\text{B.7})$$

Aging effects appear again via the multiplicative correction in the square brackets, which agrees with the result derived in [13].

In the magnetized case, the first term of (B.1) is still of $\mathcal{O}(t_w^{(2-d)/2})$ and given by $L_{\text{eq}}^{(2)}(t - t_w)(u + 1)/(ux + 1)$. The second term, on the other hand, has a leading $\mathcal{O}(t_w^{-1})$ contribution of $(1/\mu_d t)(u/t_w)(u + 1)/(ux + 1)^2 = (1/\mu_d t)ux(u + 1)/(ux + 1)^2$. The subleading terms in N_{eq} , g and g' all give corrections to this of relative order $t_w^{(4-d)/2}$ which compete with the first term of (B.1). The overall result can be written in the form

$$L^{(2)}(t, t_w) = L_{\text{eq}}^{(2)}(t - t_w)\mathcal{F}_L(x, u) + \frac{1}{\mu_d t} \frac{ux(u + 1)}{(ux + 1)^2}. \quad (\text{B.8})$$

In the aging regime ($x > 1$) the second term dominates; for $u \rightarrow \infty$ it reduces to $1/(\mu_d t x) = t_w/(\mu_d t^2)$ consistent with the result of [13]. The full expression for the aging correction factor $\mathcal{F}_L(x, u)$ in the first term is rather long so we omit it. At any rate, one sees that this first term becomes subleading compared to the second one already for time differences $t - t_w \sim [t_w(u + 1)/u]^{2/(d-2)} \ll t_w$ where $\mathcal{F}_L(x, u) = \mathcal{F}_L(1, u) = 1$. The detailed form of the multiplicative aging correction therefore never becomes relevant.

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